

LOCAL-IN-TIME EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE PRANDTL EQUATIONS BY ENERGY METHODS

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ABSTRACT. We prove local existence and uniqueness for the two-dimensional Prandtl system in weighted Sobolev spaces under the Oleinik's monotonicity assumption. In particular we do not use the Crocco transform. Our proof is based on a *new nonlinear energy estimate* for the Prandtl system. This new energy estimate is based on a cancellation property which is valid under the monotonicity assumption. To construct the solution, we use a regularization of the system that preserves this nonlinear structure. This new nonlinear structure may give some insight on the convergence properties from Navier-Stokes system to Euler system when the viscosity goes to zero.

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1. INTRODUCTION

The zero-viscosity limit of the incompressible Navier-Stokes system in a bounded domain, with Dirichlet boundary conditions, is one of the most challenging open problems in Fluid Mechanics. This is due to the formation of a boundary layer which appears because we can not impose the same Dirichlet boundary condition for the Euler equation. This boundary layer satisfies formally the Prandtl system. Indeed, in 1904, Prandtl [25] suggested that there exists a thin layer called boundary layer, where the solution \vec{u} undergoes a sharp transition from a solution to the Euler system to the no-slip boundary condition $\vec{u} = \vec{0}$ on $\partial\Omega$ of the Navier-Stokes system. In other words, Prandtl proved formally that the solution \vec{u} of the Navier-Stokes system can be written as $\vec{u} = \vec{U} + \vec{u}_{BL}$ where \vec{U} solves the Euler system with $\vec{U} \cdot \mathbf{n} = 0$ on the boundary and \vec{u}_{BL} is small except near the boundary. In rescaled variables $\vec{U} + \vec{u}_{BL}$ solves the Prandtl system. When studying this problem, there are at least 3 main questions:

- (a) The local well-posedness of the Prandtl system;
- (b) Proving the convergence of solutions of the Navier-Stokes system towards a solution of the Euler system;
- (c) The justification of the boundary layer expansion.

In full generality, these questions are still open except in the analytic case where (a)-(c) can be proved [27, 26, 16, 15].

Concerning (a), the main existence result is due to Oleinik who proved the local existence for the Prandtl system [22, 23] under a monotonicity assumption and using the Crocco transform (see also [24]). These solutions can be extended as global weak solutions if the pressure gradient is favorable ($\partial_x p \leq 0$) [33, 34]. However, E and Engquist [6] proved a blow up result for the Prandtl system for some special type of initial data. More recently, Gérard-Varet and Dormy [7] proved ill-posedness for the linearized Prandtl equation around a nonmonotonic shear flow (see also [10, 8]).

Concerning (b), the main result is a convergence criterion due to Kato [13] that basically says that convergence is equivalent to the fact that there is no dissipation in a very thin layer (of size ν). This criterion was extended in different directions (see [29, 31, 14]). Also,

in [17], it is proved that the convergence holds if the horizontal viscosity goes to zero slower than the vertical one. It is worth noting that the Prandtl system is the same in this case.

Concerning (c), there is a negative result by Grenier [9] who proves that the expansion does not hold in $W^{1,\infty}$. Of course this does not prevent (b) from holding.

There are many review papers about the inviscid limit of the Navier-Stokes in a bounded domain and the Prandtl system from different aspects (see [4, 5, 18]). Let us also mention that when considered in the whole space [28, 12, 19] or with other boundary conditions such as Navier boundary condition [32, 11, 2, 20] or incoming flow [30], the convergence problem becomes simpler since there is no boundary layer or the boundary layer is stable.

The prime objective of this paper is to prove the local existence and uniqueness for the two-dimensional Prandtl system under the Oleinik's monotonicity assumption in certain weighted energy spaces without using the Crocco transform. Precise statement will be provided in section 2. In addition to giving a very simple understanding of the monotonicity assumptions, our result may give us a better understanding about the questions (b) and (c) since it is given in physical space. Nevertheless, we are still not able to use our new nonlinear energy to study the convergence problem (b) or (c). In spirit this paper is similar to our previous paper about the Hydrostatic Euler equations [21] where we gave a proof of existence and uniqueness in physical space under some convexity assumption of the profile. The previous known proof of Brenier [3] uses Lagrangian coordinates and requires more assumptions on the initial data.

Let us end this introduction by outlining the structure of this paper. In section 2 we will state our main result, that is, theorem 2.2. Explanations of our approach and approximate scheme will be provided in sections 3 and 4. Assuming the solvability of approximate systems, we will derive our new weighted a priori estimates in section 5. Using these weighted estimates, we will complete the proof of our main theorem 2.2 in section 6. In section 7 we will solve the approximate systems. For the sake of self-containedness, we will also provide several elementary proofs and computations in appendices A - E. Finally, let us mention that the new preprint [1] also considers the existence for the Prandtl system in physical space. The methods of proof are very different.

2. MAIN RESULT

In this section we will first introduce the Prandtl equations, and then describe our solution spaces as well as our main result. Main difficulties and brief explanation of our approach will be given in section 3.

Throughout this paper, we are concerned with the two-dimensional Prandtl equations in a periodic domain $\mathbb{T} \times \mathbb{R}^+ := \{(x, y); x \in \mathbb{R}/\mathbb{Z}, 0 \leq y < +\infty\}$:

$$(2.1) \quad \begin{cases} \partial_t u + u \partial_x u + v \partial_y u = \partial_y^2 u - \partial_x p & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u + \partial_y v = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u|_{y=0} = v|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u(t, x, y) = U & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{cases}$$

where the velocity field $(u, v) := (u(t, x, y), v(t, x, y))$ is an unknown, the initial data $u_0 := u_0(x, y)$ and the outer flow $U := U(t, x)$ are given and satisfy the compatibility conditions:

$$(2.2) \quad u_0|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u_0 = U.$$

Furthermore, the given scalar pressure $p := p(t, x)$ and the outer flow U satisfy the well-known Bernoulli's law:

$$(2.3) \quad \partial_t U + U \partial_x U = -\partial_x p.$$

In this work, we will consider system (2.1) under the Oleinik's monotonicity assumption:

$$(2.4) \quad \omega := \partial_y u > 0.$$

Under this hypothesis, one must further assume $U > 0$.

Let us first introduce the function space in which the Prandtl equations (2.1) will be solved. Denoting the vorticity $\omega := \partial_y u$, we define the space $H_{\sigma, \delta}^{s, \gamma}$ for ω by

$$H_{\sigma, \delta}^{s, \gamma} := \left\{ \omega : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}; \|\omega\|_{H^{s, \gamma}} < +\infty, (1+y)^\sigma \omega \geq \delta \text{ and } \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega|^2 \leq \frac{1}{\delta^2} \right\}$$

where $s \geq 4$, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, $\delta \in (0, 1)$, $D^\alpha := \partial_x^{\alpha_1} \partial_y^{\alpha_2}$ and the weighted H^s norm $\|\cdot\|_{H^{s, \gamma}}$ is defined by

$$(2.5) \quad \|\omega\|_{H^{s, \gamma}(\mathbb{T} \times \mathbb{R}^+)}^2 := \sum_{|\alpha| \leq s} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2.$$

Here, the main idea is adding an extra weight $(1+y)$ for each y -derivative. This corresponds to the weight $\frac{1}{y}$ in the Hardy type inequality. Furthermore, we also denote $H^{s, \gamma} := \{\omega : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}; \|\omega\|_{H^{s, \gamma}} < +\infty\}$.

Remark 2.1 (Requirement: $\sigma > \gamma + \frac{1}{2}$). If $\sigma \leq \gamma + \frac{1}{2}$, then one may check that $H_{\sigma, \delta}^{s, \gamma}(\mathbb{T} \times \mathbb{R}^+)$ is an empty set. Thus, we must have the hypothesis $\sigma > \gamma + \frac{1}{2}$.

Now, we can state our main result:

Theorem 2.2 (Local $H_{\sigma,\delta}^{s,\gamma}$ Existence and Uniqueness to the Prandtl Equations (2.1)). *Let $s \geq 4$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, \frac{1}{2})$. For simplicity¹, we suppose that the outer flow U satisfies*

$$(2.6) \quad \sup_t \|U\|_{s+9,\infty} := \sup_t \sum_{l=0}^{\lfloor \frac{s+9}{2} \rfloor} \|\partial_t^l U\|_{W^{s-2l+9,\infty}(\mathbb{T})} < +\infty.$$

Assume that $u_0 - U \in H^{s,\gamma-1}$ and the initial vorticity $\omega_0 := \partial_y u_0 \in H_{\sigma,2\delta}^{s,\gamma}$. In addition, when $s = 4$, we further assume that $\delta > 0$ is chosen small enough such that

$$(2.7) \quad \|\omega_0\|_{H_g^{s,\gamma}} \leq C\delta^{-1}$$

where the norm $\|\cdot\|_{H_g^{s,\gamma}}$ will be defined by (3.1) and C is a universal constant. Then there exist a time $T := T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}}, U) > 0$ and a unique classical solution (u, v) to the Prandtl equations (2.1) such that $u - U \in L^\infty([0, T]; H^{s,\gamma-1}) \cap C([0, T]; H^s - w)$ and the vorticity $\omega := \partial_y u \in L^\infty([0, T]; H_{\sigma,\delta}^{s,\gamma}) \cap C([0, T]; H^s - w)$, where $H^s - w$ is the space H^s endowed with its weak topology.

Remark 2.3 ($U \equiv \text{constant}$). When the outer flow U is a constant, one may show that the life-span T stated in theorem 2.2 is independent of U . For the reasoning, see remark 6.3.

The proof of theorem 2.2 is based on our new weighted energy estimates, which relies on a nonlinear cancelation property that holds under the Oleinik's monotonicity assumption (2.4). An outline of our proof will be given in sections 3 and 4, and the detailed analysis will be provided in sections 5 - 7.

Before we proceed, let us comment on our notation. Throughout this paper, all constants C may be different in different lines. Subscript(s) of a constant illustrates the dependence of the constant, for example, C_s is a constant depending on s only.

3. DIFFICULTIES AND OUTLINE OF OUR APPROACH

The aim of this section is to explain main difficulties of proving theorem 2.2 as well as our strategies for overcoming them. Let us begin by stating the main difficulties as follows.

In order to solve the Prandtl equations (2.1) in certain H^s spaces, we have to overcome the following three difficulties:

- (i) the vertical velocity $v := -\partial_y^{-1} \partial_x u$ creates a loss of x -derivative, so the standard energy estimates do not apply;

¹The regularity hypothesis on the outer flow U is obviously not optimal in the viewpoint of our a priori weighted energy estimates. One may further loosen the regularity requirement on U by applying other approximate schemes. We leave this for the interested reader.

- (ii) the unboundedness of the underlying physical domain $\mathbb{T} \times \mathbb{R}^+$ allows certain quantities growth at $y = +\infty$ even if the solution is smooth or bounded in H^s ;
- (iii) the lack of higher order boundary conditions at $y = 0$ prevents us to apply the integration by parts in the variable y , but it is a standard and crucial step to deal with the operator $\partial_t - \partial_y^2$.

Indeed, difficulty (i) is the major problem for the Prandtl equations (2.1), and it explains why there are just a few existence results in the literature. The key ingredient of the current work is to develop a H^s control by considering a special H^s norm (see (3.1) below) which can avoid the regularity loss created by v . Difficulty (ii) is somewhat based on the fact that Poincaré inequality does not hold for the unbounded domain $\mathbb{T} \times \mathbb{R}^+$. However, one may overcome this technical problem by replacing the Poincaré type inequalities by Hardy type inequalities. This is our main reason for adding a weight $(1 + y)$ for each y -derivative to our H^s energies (2.5) and (3.1). Difficulty (iii) seems to be an obstacle, but it is not. A reconstruction argument for the higher order boundary conditions can fix this technical difficulty when s is even, see lemma 5.9 for more details.

Now, let us explain our new weighted energy, which is the main novelty in this paper.

Judging from nonlinear cancelations, the weighted norm (2.5) is not suitable for estimating solutions of the Prandtl equations (2.1). Thus, we introduce another weighted norm for the vorticity ω , namely

$$(3.1) \quad \|\omega\|_{H_g^{s,\gamma}(\mathbb{T} \times \mathbb{R}^+)}^2 := \|(1+y)^\gamma g_s\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}^2$$

where

$$g_s := \partial_x^s \omega - \frac{\partial_y \omega}{\omega} \partial_x^s (u - U) \quad \text{and} \quad u(t, x, y) := \int_0^y \omega(t, x, \tilde{y}) d\tilde{y}$$

provided that $\omega := \partial_y u > 0$. The difference between norms (2.5) and (3.1) is that we replace the weighted L^2 norm of $\partial_x^s \omega$ by that of g_s , which is a better quantity because g_s can avoid the loss of x -derivative (i.e., difficulty (i) above), see subsection 5.1.2 for further explanation.

The first important observation is that as long as $\omega \in H_{\sigma,\delta}^{s,\gamma}$, we can show that the new weighted norm (3.1) is almost equivalent to the weighed H^s norm (2.5), that is,

$$(3.2) \quad \|\omega\|_{H_g^{s,\gamma}} \lesssim \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \lesssim \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}$$

provided that $\omega = \partial_y u$, $u|_{y=0} = 0$ and $\lim_{y \rightarrow +\infty} u = U$. The proof of (3.2) is elementary, and will be given in appendix A. In spirit of (3.2), we will estimate $\|\omega\|_{H_g^{s,\gamma}}$ instead of $\|\omega\|_{H^{s,\gamma}}$.

The second important observation is that due to the nonlinear cancelation, the loss of x -derivative is avoided by the norm $\|\cdot\|_{H_g^{s,\gamma}}$, so one can simply derive a priori energy estimates on ω by applying the standard energy methods. These estimates indeed can be extended to $\omega^\epsilon := \partial_y u^\epsilon$, which is the regularized vorticity of the regularized Prandtl equations (4.1) below,

because the regularization (4.1) preserves the nonlinear structure of the original Prandtl equations (2.1). See subsection 5.1 for the detailed analysis.

Once we have obtained the weighted energy estimates, it remains to derive weighted L^∞ controls on the lower order derivatives of ω so that we can close our estimates in the function space $H_{\sigma,\delta}^{s,\gamma}$. The derivations of these L^∞ estimates are standard: “viewing” the evolution equations of the lower order derivatives as “linear” parabolic equations with coefficients involving higher order terms that can be bounded by the weighted energies, we can obtain our desired estimates by the classical maximum principle since we have already controlled the weighted energies. These weighted L^∞ estimates are also extendable to the regularized vorticity ω^ϵ , see subsection 5.2 for further details.

In order to prove the existence, we will construct an approximate scheme which keeps the a priori estimates described above. Due to the nonlinear cancelation, our a priori estimates are complicated in certain sense, so the construction of the approximate scheme is tricky. An outline of this construction will be given in section 4.

For the uniqueness, it is an immediate consequence of a L^2 comparison principle (see proposition 6.4), whose proof relies on a nonlinear cancelation that is similar to the one applied in the energy estimates.

4. APPROXIMATE SCHEME

The main purpose of this section is outlining the approximate systems which we apply to prove the existence. Since our weighted H^s a priori estimates are somewhat more nonlinear than usual, the approximate scheme is slightly more complicated.

Our approximate scheme has three different levels and will be explained as follows.

The first approximation of (2.1) is the regularized Prandtl equations: for any $\epsilon > 0$,

$$(4.1) \quad \left\{ \begin{array}{ll} \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon = \epsilon^2 \partial_x^2 u^\epsilon + \partial_y^2 u^\epsilon - \partial_x p^\epsilon & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0 & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ u^\epsilon|_{t=0} = u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ u^\epsilon|_{y=0} = v^\epsilon|_{y=0} = 0 & \text{on } [0, T] \times \mathbb{T} \\ \lim_{y \rightarrow +\infty} u^\epsilon(t, x, y) = U & \text{for all } (t, x) \in [0, T] \times \mathbb{T}, \end{array} \right.$$

where p^ϵ and U satisfy a regularized Bernoulli's law:

$$(4.2) \quad \partial_t U + U \partial_x U = \epsilon^2 \partial_x^2 U - \partial_x p^\epsilon.$$

Or equivalently, the regularized vorticity $\omega^\epsilon := \partial_y u^\epsilon$ satisfies the following regularized vorticity system: for any $\epsilon > 0$,

$$(4.3) \quad \begin{cases} \partial_t \omega^\epsilon + u^\epsilon \partial_x \omega^\epsilon + v^\epsilon \partial_y \omega^\epsilon = \epsilon^2 \partial_x^2 \omega^\epsilon + \partial_y^2 \omega^\epsilon & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \omega^\epsilon|_{t=0} = \omega_0 := \partial_y u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ \partial_y \omega^\epsilon|_{y=0} = \partial_x p^\epsilon & \text{on } [0, T] \times \mathbb{T} \end{cases}$$

where the velocity field (u^ϵ, v^ϵ) is given by

$$(4.4) \quad u^\epsilon(t, x, y) := U - \int_y^{+\infty} \omega^\epsilon(t, x, \tilde{y}) d\tilde{y} \quad \text{and} \quad v^\epsilon(t, x, y) := - \int_0^y \partial_x u^\epsilon(t, x, \tilde{y}) d\tilde{y}.$$

The main idea of this approximation is adding the viscous terms $\epsilon^2 \partial_x^2 u^\epsilon$ and $\epsilon^2 \partial_x^2 \omega^\epsilon$ to avoid the loss of x -derivative. The advantage of this regularization is that our new weighted H^s and L^∞ a priori estimates also hold for ω^ϵ , and it is the main reason why we can derive the uniform (in ϵ) estimates in section 5. The price that we pay is the appearance of extra terms $\frac{\partial_x \omega^\epsilon}{\omega^\epsilon}, \frac{\partial_x^2 \omega^\epsilon}{\omega^\epsilon}, \frac{\partial_x \partial_y \omega^\epsilon}{\omega^\epsilon}$ and $\frac{\partial_y^2 \omega^\epsilon}{\omega^\epsilon}$ during the estimation, but these terms can be controlled in the function space $C([0, T]; H_{\sigma, \delta}^{s, \gamma})$. Before going to the next level, we should also emphasize that replacing the Bernoulli's law (2.3) by the regularized Bernoulli's law (4.2) is crucial here, otherwise the conditions $u|_{y=0} = 0$ and $\lim_{y \rightarrow +\infty} u = U$ cannot be satisfied simultaneously. Although the approximate system (4.3) - (4.4) seems to be nice, its existence in the function space $H_{\sigma, \delta}^{s, \gamma}$ is not obvious, so we will further approximate it by the next approximate system.

The second level of approximation is the truncated and regularized vorticity system: for any $\epsilon > 0$ and $R \geq 1$,

$$(4.5) \quad \begin{cases} \partial_t \omega_R + \chi_R \{u_R \partial_x \omega_R + v_R \partial_y \omega_R\} = \epsilon^2 \partial_x^2 \omega_R + \partial_y^2 \omega_R & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \omega_R|_{t=0} = \omega_0 := \partial_y u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ \partial_y \omega_R|_{y=0} = \partial_x p^\epsilon & \text{on } [0, T] \times \mathbb{T} \end{cases}$$

where the velocity field (u_R, v_R) is given by

$$(4.6) \quad u_R(t, x, y) := U - \int_y^{+\infty} \omega_R(t, x, \tilde{y}) d\tilde{y} \quad \text{and} \quad v_R(t, x, y) := - \int_0^y \partial_x u_R(t, x, \tilde{y}) d\tilde{y}.$$

Here, p^ϵ and U still satisfy the regularized Bernoulli's law (4.2). The cutoff function χ_R is defined by $\chi_R(y) := \chi(\frac{y}{R})$ where $\chi \in C_c^\infty([0, +\infty))$ satisfies the following properties:

$$(4.7) \quad 0 \leq \chi \leq 1, \quad \chi|_{[0, 1]} \equiv 1, \quad \text{supp } \chi \subseteq [0, 2] \quad \text{and} \quad -2 \leq \chi' \leq 0.$$

The main disadvantage of approximate system (4.5) - (4.6) is that the truncation on the convection term $u_R \partial_x \omega_R + v_R \partial_y \omega_R$ destroys the boundary condition $u_R|_{y=0} = 0$ as well as our weighted H^s a priori estimate. However, it still keeps the weighted L^∞ controls.

To compensate for the lack of our new weighted H^s estimates, one may apply the standard H^s energy estimates because the system (4.5) - (4.6) does not have the problem of x -derivative loss. These estimates depend on ϵ , but not on R . Thus, passing to the limit $R \rightarrow +\infty$ for the solution of (4.5) - (4.6) to that of (4.3) - (4.4) should be no doubt. The reason of doing this approximation is to prepare for our next approximate system.

The third level of approximation is the linearized, truncated and regularized vorticity system: for any $\epsilon > 0$, $R \geq 1$ and $n \in \mathbb{N}$,

$$(4.8) \quad \begin{cases} \partial_t \omega^{n+1} + \chi_R \{u^n \partial_x \omega^n + v^n \partial_y \omega^n\} = \epsilon^2 \partial_x^2 \omega^{n+1} + \partial_y^2 \omega^{n+1} & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ \omega^{n+1}|_{t=0} = \omega_0 := \partial_y u_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ \partial_y \omega^{n+1}|_{y=0} = \partial_x p^\epsilon & \text{on } [0, T] \times \mathbb{T} \end{cases}$$

where the velocity field (u^n, v^n) is given by

$$(4.9) \quad u^n(t, x, y) := U - \int_y^{+\infty} \omega^n(t, x, \tilde{y}) d\tilde{y} \quad \text{and} \quad v^n(t, x, y) := - \int_0^y \partial_x u^n(t, x, \tilde{y}) d\tilde{y}.$$

In other words, (4.8) - (4.9) is a linearization of (4.5) - (4.6).

The main advantage of the iterative scheme (4.8) - (4.9) is that its explicit solution formula can be obtained by the method of reflection. Using the explicit solution formula and the fact that $\chi_R \{u^n \partial_x u^n + v^n \partial_y u^n\}$ has compact support, one may prove that there exists a uniform (in n) life-span $T > 0$ for the approximate sequence $\{\omega^n\}_{n \in \mathbb{N}}$ provided that $\omega_0 \in H_{\sigma, 2\delta}^{s, \gamma}$. This gives us a starting point so that we can solve the approximate systems and derive estimates.

Solving the above approximate systems in a reverse order and deriving appropriate estimates, we can prove the existence to the Prandtl equations (2.1). Detailed analysis for solving the regularization (4.1) as well as other approximate systems (4.3) - (4.4), (4.5) - (4.6) and (4.8) - (4.9) will be given in section 7. Assuming that $(u^\epsilon, v^\epsilon, \omega^\epsilon)$ solves (4.1) - (4.4), we will derive uniform (in ϵ) weighted estimates in section 5. Based on these uniform estimates, we will complete the proof of our main theorem 2.2 in section 6.

5. UNIFORM ESTIMATES ON THE REGULARIZED PRANDTL EQUATIONS

In this section and the next we are going to complete the proof of our main theorem 2.2 provided that we have a solution of the regularized Prandtl equations (4.1). In this section we will derive uniform estimates for the regularized Prandtl equations (4.1) by using the new weighted energy (3.1) introduced in section 3. These estimates are the main novelty of this paper. Then we will finish the proof of theorem 2.2 in section 6. After that, an outline for solving the regularized Prandtl equations (4.1) will be provided in section 7.

Our starting point is that we can solve the velocity field (u^ϵ, v^ϵ) from the regularized Prandtl equations (4.1). More precisely, let us assume proposition 5.1, which will be shown in section 7, below for the moment.

Proposition 5.1 (Local Existence of the Regularized Prandtl Equations). *Let $s \geq 4$ be an even integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, \frac{1}{2})$ and $\epsilon \in (0, 1]$. If $\omega_0 \in H_{\sigma, 2\delta}^{s+12, \gamma}, U$ and p^ϵ are given and satisfy the regularized Bernoulli's law (4.2) and the regularity assumption (2.6), then there exist a time $T := T(s, \gamma, \sigma, \delta, \epsilon, \|\omega_0\|_{H^{s+4, \gamma}}, U) > 0$ and a solution $\omega^\epsilon \in C([0, T]; H_{\sigma, \delta}^{s+4, \gamma}) \cap C^1([0, T]; H^{s+2, \gamma})$ to the regularized vorticity system (4.3)-(4.4).*

Furthermore, the velocity (u^ϵ, v^ϵ) defined by (4.4) satisfies the regularized Prandtl equations (4.1) as well.

Remark 5.2 (Initial Data). The $H_{\sigma, 2\delta}^{s, \gamma}$ functions can be approximated by $H_{\sigma, 2\delta}^{s+12, \gamma}$ functions in the norm $\|\cdot\|_{H^{s, \gamma}}$, so by the standard density argument, the hypothesis $\omega_0 \in H_{\sigma, 2\delta}^{s+12, \gamma}$ can be reduced to be $\omega_0 \in H_{\sigma, 2\delta}^{s, \gamma}$ in our final result.

According to proposition 5.1, the life-span $T_{s, \gamma, \sigma, \delta, \epsilon, \omega_0, U}$ of ω^ϵ depends on ϵ , so our aim in this section is to remove the ϵ -dependence by deriving uniform (in ϵ) estimates on ω^ϵ . In other words, we will prove the following

Proposition 5.3 (Uniform Estimates on the Regularized Prandtl Equations). *Let $s \geq 4$ be an even integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, 1)$ and $\epsilon \in [0, 1]$. If $\omega^\epsilon \in C([0, T]; H_{\sigma, \delta}^{s+4, \gamma}) \cap C^1([0, T]; H_{\sigma, \delta}^{s+2, \gamma})$ and $(u^\epsilon, v^\epsilon, \omega^\epsilon)$ solves (4.1) - (4.4), then we have the following uniform (in ϵ) estimates:*

(i) (Weighted H^s Estimates)

$$(5.1) \quad \begin{aligned} & \|\omega^\epsilon(t)\|_{H_g^{s, \gamma}} \\ & \leq \left\{ \|\omega_0\|_{H_g^{s, \gamma}}^2 + \int_0^t F(\tau) d\tau \right\}^{\frac{1}{2}} \left\{ 1 - C_{s, \gamma, \sigma, \delta} \left(\|\omega_0\|_{H_g^{s, \gamma}}^2 + \int_0^t F(\tau) d\tau \right)^{\frac{s-2}{2}} t \right\}^{-\frac{1}{s-2}} \end{aligned}$$

as long as the second braces on the right hand side of (5.1) is positive, where the positive constant $C_{s, \gamma, \sigma, \delta}$ depends on $s, \gamma, \sigma, \delta$ only and $F : [0, T] \rightarrow \mathbb{R}^+$ is defined by

$$(5.2) \quad F := C_{s, \gamma, \sigma, \delta} \{1 + \|\partial_x^{s+1} U\|_{L^\infty}^4\} + C_s \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l \partial_x p^\epsilon\|_{H^{s-2l}(\mathbb{T})}^2.$$

(ii) (Weighted L^∞ Estimates) Define $I(t) := \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega^\epsilon(t)|^2$. For any $s \geq 4$,

$$(5.3) \quad \|I(t)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq \max\{\|I(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, 6C^2 \Omega(t)^2\} e^{C_{s, \gamma, \sigma, \delta} \{1+G(t)\}t}$$

where the universal constant C is the same as the one in inequality (B.3), Ω and $G : [0, T] \rightarrow \mathbb{R}^+$ are defined by

$$(5.4) \quad \Omega(t) := \sup_{[0, t]} \|\omega^\epsilon\|_{H_g^{s, \gamma}} \quad \text{and} \quad G(t) := \sup_{[0, t]} \|\omega^\epsilon\|_{H_g^{s, \gamma}} + \sup_{[0, t]} \|\partial_x^s U\|_{L^2(\mathbb{T})}.$$

In addition, if $s \geq 6$, then we also have

$$(5.5) \quad \|I(t)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq (\|I(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} + C_{s,\gamma}\{1 + \Omega(t)\}\Omega(t)^2 t)e^{C_{s,\gamma,\sigma,\delta}\{1+G(t)\}t}.$$

For $s \geq 4$, we have the following lower bound estimate:

$$(5.6) \quad \begin{aligned} & \min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega^\epsilon(t) \\ & \geq (1 - C_{s,\gamma,\sigma,\delta}\{1 + G(t)\}te^{C_{s,\gamma,\sigma,\delta}\{1+G(t)\}t}) \cdot \left(\min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega_0 - C_{s,\gamma}\Omega(t)t \right) \end{aligned}$$

provided that $\min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega_0 - C_{s,\gamma}\Omega(t)t \geq 0$, where $C_{s,\gamma,\sigma,\delta}$ is a positive constant depending on s, γ, σ and δ only.

Remark 5.4 (Two L^∞ Estimates on I). In proposition 5.3, we stated two L^∞ controls on the quantity $I(t)$, namely, estimates (5.3) and (5.5). Indeed, (5.5) is a better estimate within a short time, but it only holds for $s \geq 6$. Thanks to this better estimate, we can derive the uniform weighted L^∞ bound (6.2) without any additional assumption when $s \geq 6$. In contrast, we are required to impose an extra initial hypothesis (2.7) for the case $s = 4$ since we only have the weaker estimate (5.3) in this case. See proposition 6.1 for the details.

Remark 5.5 (A Priori Estimates on the Prandtl Equations). When $\epsilon = 0$, proposition 5.3 provides a priori estimates for the Prandtl equations (2.1). Similar situation occurs in proposition 6.1 as well.

The proof of proposition 5.3 will be given in the subsections 5.1 and 5.2 as follows.

5.1. Weighted Energy Estimates. The objective of this subsection is to derive uniform (in ϵ) weighted H^s estimates on ω^ϵ . These estimates, which are the main novelty of this paper, include: (i) weighted L^2 estimates on $D^\alpha \omega^\epsilon$ for $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$ in subsubsection 5.1.1, and (ii) weighted L^2 estimate on g_s in subsubsection 5.1.2. We will combine these two estimates in subsubsection 5.1.3 to obtain the uniform weighted energy estimates (5.1). This will complete the proof of part (i) of proposition 5.3.

5.1.1. Weighted L^2 Estimates on $D^\alpha \omega^\epsilon$. Using the standard energy method, we will derive weighted L^2 estimates on $D^\alpha \omega^\epsilon$ for $|\alpha| \leq s$ and $\alpha_1 \leq s - 1$ in this subsubsection. It works because we are allowed to loss at least one x -regularity in these cases.

More specifically, we will prove

Proposition 5.6 (L^2 Controls on $(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon$ for $|\alpha| \leq s$ and $\alpha_1 \leq s-1$). *Under the hypotheses of proposition 5.3, we have the following estimates:*

$$\begin{aligned}
(5.7) \quad & \frac{1}{2} \frac{d}{dt} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2}^2 \\
& \leq -\epsilon^2 \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega^\epsilon\|_{L^2}^2 - \frac{1}{2} \sum_{\substack{|\alpha| \leq s \\ \alpha_1 \leq s-1}} \|(1+y)^{\gamma+\alpha_2} \partial_y D^\alpha \omega^\epsilon\|_{L^2}^2 \\
& \quad + C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 \\
& \quad + C_{s,\gamma,\sigma,\delta} \{ 1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}} \}^{s-2} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 + C_s \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l \partial_x p^\epsilon\|_{H^{s-2l}(\mathbb{T})}^2,
\end{aligned}$$

where the positive constants C_s and $C_{s,\gamma,\sigma,\delta}$ are independent of ϵ .

Remark 5.7 (Boundary Terms at $y = +\infty$). In the proof of proposition 5.6 and that of proposition 5.10 below, we will ignore the boundary terms at $y = +\infty$ while we are integrating by parts in the variable y . Skipping these boundary terms is just for the presentation convenience, and ignoring these technicalities is harmless. Indeed, one may deal with these boundary terms by any one of the following two methods:

- (i) Since $\omega^\epsilon \in H_{\sigma,\delta}^{s+4,\gamma}$, by proposition C.1, we have nice pointwise decays (C.1) for ω^ϵ and its spatial derivatives. Therefore, when σ is much larger than γ , one may easily check that those terms which we will omit actually vanish;
- (ii) As long as $\omega^\epsilon(t) \in H_{\sigma,\delta}^{s,\gamma}$, the norm $\|\omega^\epsilon\|_{H_g^{s,\gamma}} < +\infty$ provides certain integrability of the underlying quantities. Thus, one may overcome the technical difficulty by first multiplying by a nice cutoff function $\chi_R(y) := \chi(\frac{y}{R})$ during the estimation, and then passing to the limit $R \rightarrow +\infty$. The main advantage of this approach is that it only requires $\sigma > \gamma + \frac{1}{2}$ and $\omega^\epsilon(t)$ in $H_{\sigma,\delta}^{s,\gamma}$, but not in $H_{\sigma,\delta}^{s+4,\gamma}$. As a demonstration, we will apply this argument in the proof of proposition 6.4 for the reader's convenience.

In conclusion, the proofs of proposition 5.6 and 5.10 are absolutely correct, even if we ignore the boundary terms at $y = +\infty$.

Proof of proposition 5.6. Differentiating the vorticity equation (4.3)₁ with respect to x α_1 times and y α_2 times, we obtain the evolution equation for $D^\alpha \omega^\epsilon$:

$$(5.8) \quad \{ \partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2 \} D^\alpha \omega^\epsilon = - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \{ D^\beta u^\epsilon \partial_x D^{\alpha-\beta} \omega^\epsilon + D^\beta v^\epsilon \partial_y D^{\alpha-\beta} \omega^\epsilon \}.$$

Multiplying (5.8) by $(1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon$, and then integrating over $\mathbb{T} \times \mathbb{R}^+$, we have

$$\begin{aligned}
(5.9) \quad & \frac{1}{2} \frac{d}{dt} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2}^2 \\
&= \epsilon^2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_x^2 D^\alpha \omega^\epsilon + \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_y^2 D^\alpha \omega^\epsilon \\
&\quad - \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \{u^\epsilon \partial_x D^\alpha \omega^\epsilon + v^\epsilon \partial_y D^\alpha \omega^\epsilon\} \\
&\quad - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \{D^\beta u^\epsilon \partial_x D^{\alpha-\beta} \omega^\epsilon + D^\beta v^\epsilon \partial_y D^{\alpha-\beta} \omega^\epsilon\}.
\end{aligned}$$

Now that we can apply integration by parts and the standard Sobolev's type estimates on trilinear forms to control the right hand side of (5.9) as follows.

Claim 5.8. *There exist constants $C_{s,\gamma}$ and $C_{s,\gamma,\sigma,\delta} > 0$ such that for any $|\alpha| \leq s$ and $\alpha_1 \leq s-1$,*

$$(5.10) \quad \epsilon^2 \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_x^2 D^\alpha \omega^\epsilon = -\epsilon^2 \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega^\epsilon\|_{L^2}^2.$$

$$\begin{aligned}
(5.11) \quad & \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_y^2 D^\alpha \omega^\epsilon \\
&\leq -\frac{3}{4} \|(1+y)^{\gamma+\alpha_2} \partial_y D^\alpha \omega^\epsilon\|_{L^2}^2 - \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx \Big|_{y=0} + C_{s,\gamma} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.
\end{aligned}$$

$$\begin{aligned}
(5.12) \quad & \left| \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \{u^\epsilon \partial_x D^\alpha \omega^\epsilon + v^\epsilon \partial_y D^\alpha \omega^\epsilon\} \right| \\
&\leq C_{s,\gamma,\sigma,\delta} \{\|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2
\end{aligned}$$

$$\begin{aligned}
(5.13) \quad & \left| \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \{D^\beta u^\epsilon \partial_x D^{\alpha-\beta} \omega^\epsilon + D^\beta v^\epsilon \partial_y D^{\alpha-\beta} \omega^\epsilon\} \right| \\
&\leq C_{s,\gamma,\sigma,\delta} \{\|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty}\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.
\end{aligned}$$

Assuming claim 5.8, which will be shown later in this section, for the moment, we can apply inequalities (5.10) - (5.13) to the equality (5.9), and obtain

$$\begin{aligned}
(5.14) \quad & \frac{1}{2} \frac{d}{dt} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2}^2 \\
&\leq -\epsilon^2 \|(1+y)^{\gamma+\alpha_2} \partial_x D^\alpha \omega^\epsilon\|_{L^2}^2 - \frac{3}{4} \|(1+y)^{\gamma+\alpha_2} \partial_y D^\alpha \omega^\epsilon\|_{L^2}^2 \\
&\quad - \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx \Big|_{y=0} + C_{s,\gamma} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 + C_{s,\gamma,\sigma,\delta} \{\|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty}\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.
\end{aligned}$$

When $|\alpha| \leq s - 1$, we can apply the simple trace estimate

$$(5.15) \quad \left| \int_{\mathbb{T}} |f| dx \right|_{y=0} \leq C \left\{ \int_0^1 \int_{\mathbb{T}} |f| dx dy + \int_0^1 \int_{\mathbb{T}} |\partial_y f| dx dy \right\}$$

to control the boundary integral $\int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx \big|_{y=0}$ as follows:

$$(5.16) \quad \left| \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx \big|_{y=0} \right| \leq \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \partial_y^2 D^\alpha \omega^\epsilon\|_{L^2}^2 + C \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.$$

However, when $|\alpha| = s$, a main difficulty arises: the order of $\partial_y D^\alpha \omega^\epsilon|_{y=0}$ is too high so that we cannot control the boundary integral $\int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx \big|_{y=0}$ by the simple trace estimate (5.15). In order to make use of (5.15), we must reduce the order of the problematic term $\partial_y D^\alpha \omega^\epsilon|_{y=0}$. When s is even, this can be done by a boundary reduction argument as follows.

At this moment, let us state without proof the following boundary reduction lemma, which will be proven at the end of this subsection.

Lemma 5.9 (Reduction of Boundary Data). *Under the hypotheses of proposition 5.3, we have at the boundary $y = 0$,*

$$(5.17) \quad \begin{cases} \partial_y \omega^\epsilon|_{y=0} = \partial_x p^\epsilon \\ \partial_y^3 \omega^\epsilon|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2) \partial_x p^\epsilon + \omega^\epsilon \partial_x \omega^\epsilon|_{y=0}. \end{cases}$$

For any $2 \leq k \leq \lfloor \frac{s}{2} \rfloor$, there are some constants $C_{k,l,\rho^1,\rho^2,\dots,\rho^j}$'s, which do not depend on ϵ or $(u^\epsilon, v^\epsilon, \omega^\epsilon)$, such that

$$(5.18) \quad \partial_y^{2k+1} \omega^\epsilon|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^k \partial_x p^\epsilon + \sum_{l=0}^{k-1} \epsilon^{2l} \sum_{j=2}^{\max\{2,k-l\}} \sum_{\rho \in A_{k,l}^j} C_{k,l,\rho^1,\rho^2,\dots,\rho^j} \prod_{i=1}^j D^{\rho^i} \omega^\epsilon|_{y=0}$$

where $A_{k,l}^j := \{\rho := (\rho^1, \rho^2, \dots, \rho^j) \in \mathbb{N}^{2j}; 3 \sum_{i=1}^j \rho_1^i + \sum_{i=1}^j \rho_2^i = 2k + 4l + 1, \sum_{i=1}^j \rho_1^i \leq k + 2l - 1, \sum_{i=1}^j \rho_2^i \leq 2k - 2l - 2 \text{ and } |\rho^i| \leq 2k - l - 1 \text{ for all } i = 1, 2, \dots, j\}$.

Now, we can apply lemma 5.9 to control the boundary integral $\int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx \big|_{y=0}$ for $|\alpha| = s$ with $0 \leq \alpha_1 \leq s - 1$ in the following two cases:

Case I: (α_2 is even)

When $\alpha_2 := 2k$ for some $k \in \mathbb{N}$, we can apply boundary reduction lemma 5.9 to $\partial_y D^\alpha \omega^\epsilon|_{y=0}$, and obtain

$$\begin{aligned}
 (5.19) \quad & \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx|_{y=0} \\
 &= \int_{\mathbb{T}} D^\alpha \omega^\epsilon (\partial_t - \epsilon^2 \partial_x^2)^k \partial_x^{\alpha_1+1} p^\epsilon dx|_{y=0} \\
 &+ \sum_{l=0}^{k-1} \epsilon^{2l} \sum_{j=2}^{\max\{2, k-l\}} \sum_{\rho \in A_{k,l}^j} C_{k,l,\rho^1,\rho^2,\dots,\rho^j} \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_x^{\alpha_1} \left(\prod_{i=1}^j D^{\rho^i} \omega^\epsilon \right) dx|_{y=0}.
 \end{aligned}$$

According to the definition of $A_{k,l}^j$, one may check by using the indices restrictions that the largest possible order for $\partial_x^{\alpha_1} D^{\rho^i} \omega^\epsilon$ is $\leq s-1$ and at most one of $\partial_x^{\alpha_1} D^{\rho^i} \omega^\epsilon$ can attain the order $s-1$, namely, the orders of other terms are $\leq s-2$. Therefore, we can apply the simple trace estimate (5.15) and proposition B.3 to the identity (5.19) to obtain

$$\begin{aligned}
 (5.20) \quad & \left| \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx|_{y=0} \right| \\
 & \leq \frac{1}{12} \|(1+y)^{\gamma+\alpha_2} \partial_y D^\alpha \omega^\epsilon\|_{L^2}^2 + C_s \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l \partial_x p^\epsilon\|_{H^{s-2l}(\mathbb{T})}^2 + C_{s,\gamma,\sigma,\delta} \{1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}}\}^{s-2} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.
 \end{aligned}$$

Case II: (α_2 is odd)

When $\alpha_2 := 2k+1$ for some $k \in \mathbb{N}$, since $\alpha_1 + \alpha_2 = s$ is assumed to be even, we know that $\alpha_1 \geq 1$. Using integration by parts in x , we have

$$(5.21) \quad \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx|_{y=0} = - \int_{\mathbb{T}} \partial_x D^\alpha \omega^\epsilon \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+1} \omega^\epsilon dx|_{y=0}.$$

Now, the term $\partial_x D^\alpha \omega^\epsilon|_{y=0} = \partial_x^{\alpha_1+1} \partial_y^{2k+1} \omega^\epsilon|_{y=0}$ has an odd number of y derivatives, and hence, we can apply the boundary reduction lemma 5.9 to reduce the order of the right hand side of (5.21). Similar to the Case I, we can further apply the simple trace estimates (5.15) and proposition B.3 to eventually obtain the following estimates:

$$\begin{aligned}
 (5.22) \quad & \left| \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx|_{y=0} \right| \leq \frac{1}{12} \|(1+y)^{\gamma+\alpha_2+1} \partial_x^{\alpha_1-1} \partial_y^{\alpha_2+2} \omega^\epsilon\|_{L^2}^2 + C_s \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l \partial_x p^\epsilon\|_{H^{s-2l}(\mathbb{T})}^2 \\
 & + C_{s,\gamma,\sigma,\delta} \{1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}}\}^{s-2} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.
 \end{aligned}$$

Finally, combining estimates (5.14), (5.16), (5.20) and (5.22) and summing over α , we prove (5.7). □

In order to complete the proof of proposition 5.6, it remains to show claim 5.8 and the boundary reduction lemma 5.9. Let us first prove the claim 5.8 as follows.

Proof of claim 5.8.

Proof of (5.10):

The equality (5.10) follows immediately from an integration by parts in the variable x .

Proof of (5.11):

Integrating by parts in y (cf. remark 5.7), we have

$$\begin{aligned}
& \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_y^2 D^\alpha \omega^\epsilon \\
&= - \|(1+y)^{\gamma+\alpha_2} \partial_y D^\alpha \omega^\epsilon\|_{L^2}^2 - \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx|_{y=0} \\
&\quad - 2(\gamma + \alpha_2) \iint (1+y)^{2\gamma+2\alpha_2-1} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon \\
&\leq - \frac{3}{4} \|(1+y)^{\gamma+\alpha_2} \partial_y D^\alpha \omega^\epsilon\|_{L^2}^2 - \int_{\mathbb{T}} D^\alpha \omega^\epsilon \partial_y D^\alpha \omega^\epsilon dx|_{y=0} + C_{s,\gamma} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2,
\end{aligned}$$

which is inequality (5.11).

Proof of (5.12):

Integrating by parts (cf. remark 5.7), and using $\partial_x u^\epsilon + \partial_y v^\epsilon = 0$, we have

$$(5.23) \quad \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \{u^\epsilon \partial_x D^\alpha \omega^\epsilon + v^\epsilon \partial_y D^\alpha \omega^\epsilon\} = (\gamma + \alpha_2) \iint (1+y)^{2\gamma+2\alpha_2-1} v^\epsilon |D^\alpha \omega^\epsilon|^2.$$

which and inequality (B.9) imply inequality (5.12).

Proof of (5.13):

Using the facts that $\partial_y v^\epsilon = -\partial_x u^\epsilon$ and $\partial_y u^\epsilon = \omega^\epsilon$, one may check that all terms on the left hand side of (5.13) are one of the following three types: denoting $e_1 := (1, 0)$ and $e_2 := (0, 1)$, for $\eta \in \mathbb{N}$ and $\kappa, \theta \in \mathbb{N}^2$,

Type I:

$$J_1 := \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_x^\eta v^\epsilon D^\kappa \omega^\epsilon$$

where $1 \leq \eta \leq s-1$ and $\eta e_1 + \kappa = \alpha + e_2$,

Type II:

$$J_2 := \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon \partial_x^\eta u^\epsilon D^\kappa \omega^\epsilon$$

where $1 \leq \eta \leq s$ and $\eta e_1 + \kappa = \alpha + e_1$,

Type III:

$$J_3 := \iint (1+y)^{2\gamma+2\alpha_2} D^\alpha \omega^\epsilon D^\theta \omega^\epsilon D^\kappa \omega^\epsilon$$

where $|\theta| \leq s-1$ and $\theta + \kappa = \alpha + e_1 - e_2$.

Thus, it suffices to control J_1, J_2 and J_3 by the right hand side of (5.13) as follows.

Estimates for Type I:

When $1 \leq \eta \leq s-2$, applying proposition B.3, we have, since $\kappa_2 = \alpha_2 + 1$,

$$\begin{aligned} |J_1| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \frac{\partial_x^\eta v^\epsilon}{1+y} \right\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega^\epsilon\|_{H^{s,\gamma}}^2. \end{aligned}$$

When $\eta = s-1$, by triangle inequality and proposition B.3, we have, since $\kappa_2 = \alpha_2 + 1$,

$$\begin{aligned} |J_1| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \left\| \frac{\partial_x^{s-1} v^\epsilon + y \partial_x^s U}{1+y} \right\|_{L^2} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^\infty} \\ &\quad + \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|\partial_x^s U\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2. \end{aligned}$$

In conclusion, J_1 can be controlled by the right hand side of (5.13).

Estimates for Type II:

When $1 \leq \eta \leq s-1$, applying proposition B.3, we have, since $\kappa_2 = \alpha_2$,

$$\begin{aligned} |J_2| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|\partial_x^\eta u^\epsilon\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2. \end{aligned}$$

When $\eta = s$, by triangle inequality and proposition B.3, we have, since $\kappa = (0, \alpha_2)$ and $0 \leq \alpha_2 \leq 1$,

$$\begin{aligned} |J_2| &\leq \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|\partial_x^s (u - U)\|_{L^2} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^\infty} \\ &\quad + \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|\partial_x^s U\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2. \end{aligned}$$

In conclusion, J_2 can also be controlled by the right hand side of (5.13).

Estimates for Type III:

Applying proposition B.3, we have, since $\theta_2 + \kappa_2 = \alpha_2 - 1$,

$$\begin{aligned} |J_3| &\leq \begin{cases} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|(1+y)^{1+\theta_2} D^\theta \omega^\epsilon\|_{L^\infty} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^2} & \text{if } |\theta| \leq s-2 \\ \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega^\epsilon\|_{L^2} \|(1+y)^{1+\theta_2} D^\theta \omega^\epsilon\|_{L^2} \|(1+y)^{\gamma+\kappa_2} D^\kappa \omega^\epsilon\|_{L^\infty} & \text{if } |\theta| = s-1 \end{cases} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2. \end{aligned}$$

Combining all estimates for type I - III, we prove inequality (5.13). □

Lastly, we will prove the boundary reduction lemma 5.9 as follows.

Proof of lemma 5.9. First of all, let us mention that equality (5.17)₁ is exactly the same as the given boundary condition (4.3)₃. Furthermore, differentiating the vorticity (4.3)₁ with respect to y , and then evaluating at $y = 0$, we obtain equality (5.17)₂ by using (5.17)₁ and $u^\epsilon|_{y=0} = v^\epsilon|_{y=0} \equiv 0$. Thus, it remains to prove the formula (5.18).

In order to illustrate the idea, let us derive the formula (5.18) for the case $k = 2$ as follows.

Differentiating the vorticity equation (4.3)₁ with respect to y thrice, and then evaluating at $y = 0$, we obtain, by using (5.17)₂ and $u^\epsilon|_{y=0} = v^\epsilon|_{y=0} \equiv 0$,

$$(5.24) \quad \partial_y^5 \omega^\epsilon|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^2 \partial_x p^\epsilon + (\partial_t - \epsilon^2 \partial_x^2)(\omega^\epsilon \partial_x \omega^\epsilon) + 3\omega^\epsilon \partial_x \partial_y^2 \omega^\epsilon + 2\partial_y \omega^\epsilon \partial_x \partial_y \omega^\epsilon - 2\partial_x \omega^\epsilon \partial_y^2 \omega^\epsilon|_{y=0}.$$

Since the last three terms on the right hand side are our desired forms, we only need to deal with the terms $(\partial_t - \epsilon^2 \partial_x^2)(\omega^\epsilon \partial_x \omega^\epsilon)|_{y=0}$. Using the evolution equations for ω^ϵ and $\partial_x \omega^\epsilon$ as well as $u^\epsilon|_{y=0} = v^\epsilon|_{y=0} \equiv 0$, one may check that

$$(5.25) \quad (\partial_t - \epsilon^2 \partial_x^2)(\omega^\epsilon \partial_x \omega^\epsilon)|_{y=0} = \omega^\epsilon \partial_x \partial_y^2 \omega^\epsilon + \partial_x \omega^\epsilon \partial_y^2 \omega^\epsilon - 2\epsilon^2 \partial_x \omega^\epsilon \partial_x^2 \omega^\epsilon|_{y=0},$$

where all terms on the right hand side of (5.25) are also our desired forms. Substituting (5.25) into (5.24), we justify the formula (5.18) for $k = 2$.

Now, using the same algorithm, we are going to prove the formula (5.18) by induction on k . For the notational convenience, we denote

$$\mathcal{A}_k := \left\{ \sum_{l=0}^{k-1} \epsilon^{2l} \sum_{j=2}^{\max\{2,k-l\}} \sum_{\rho \in A_{k,l}^j} C_{k,l,\rho^1,\rho^2,\dots,\rho^j} \prod_{i=1}^j D^{\rho^i} \omega^\epsilon|_{y=0} \right\}.$$

Under this notation, we will prove $\partial_y^{2k+1} \omega^\epsilon|_{y=0} - (\partial_t - \epsilon^2 \partial_x^2)^k \partial_x p^\epsilon \in \mathcal{A}_k$.

Assuming that the formula (5.18) holds for $k = n$, we will show that it also holds for $k = n + 1$ as follows.

In order to reduce the order of $\partial_y^{2n+3}\omega^\epsilon|_{y=0}$, we first differentiate the vorticity (4.3)₁ with respect to y $2n+1$ times, and then evaluate the resulting equation at $y=0$ to obtain

$$(5.26) \quad \begin{aligned} \partial_y^{2n+3}\omega^\epsilon|_{y=0} &= (\partial_t - \epsilon^2\partial_x^2)\partial_y^{2n+1}\omega^\epsilon + \sum_{m=1}^{2n+1} \binom{2n+1}{m} \partial_y^{m-1}\omega^\epsilon \partial_x \partial_y^{2n-m+1}\omega^\epsilon \\ &\quad - \sum_{m=2}^{2n+1} \binom{2n+1}{m} \partial_x \partial_y^{m-2}\omega^\epsilon \partial_y^{2n-m+2}\omega^\epsilon|_{y=0}. \end{aligned}$$

By a routine checking, one may show that the last two terms of (5.26) belong to \mathcal{A}_{n+1} , so it remains to deal with $(\partial_t - \epsilon^2\partial_x^2)\partial_y^{2n+1}\omega^\epsilon|_{y=0}$ only.

Thanks to the induction hypothesis, there exist constants $C_{n,l,\rho^1,\rho^2,\dots,\rho^j}$'s such that

$$\partial_y^{2n+1}\omega^\epsilon|_{y=0} = (\partial_t - \epsilon^2\partial_x^2)^n \partial_x p^\epsilon + \sum_{l=0}^{n-1} \epsilon^{2l} \sum_{j=2}^{\max\{2,n-l\}} \sum_{\rho \in A_{n,l}^j} C_{n,l,\rho^1,\rho^2,\dots,\rho^j} \prod_{i=1}^j D^{\rho^i} \omega^\epsilon|_{y=0},$$

so we have, up to a relabeling of the indices ρ^i 's,

$$(5.27) \quad \begin{aligned} &(\partial_t - \epsilon^2\partial_x^2)\partial_y^{2n+1}\omega^\epsilon|_{y=0} \\ &= (\partial_t - \epsilon^2\partial_x^2)^{n+1} \partial_x p^\epsilon + \sum_{l=0}^{n-1} \epsilon^{2l} \sum_{j=2}^{\max\{2,n-l\}} \sum_{\rho \in A_{n,l}^j} \tilde{C}_{n,l,\rho^1,\rho^2,\dots,\rho^j} (\partial_t - \epsilon^2\partial_x^2) D^{\rho^1} \omega^\epsilon \prod_{i=2}^j D^{\rho^i} \omega^\epsilon \\ &\quad - \sum_{l=0}^{n-1} \epsilon^{2l+2} \sum_{j=2}^{\max\{2,n-l\}} \sum_{\rho \in A_{n,l}^j} \tilde{\tilde{C}}_{n,l,\rho^1,\rho^2,\dots,\rho^j} \partial_x D^{\rho^1} \omega^\epsilon \partial_x D^{\rho^2} \omega^\epsilon \prod_{i=3}^j D^{\rho^i} \omega^\epsilon|_{y=0} \end{aligned}$$

where $\tilde{C}_{n,l,\rho^1,\rho^2,\dots,\rho^j}$'s and $\tilde{\tilde{C}}_{n,l,\rho^1,\rho^2,\dots,\rho^j}$'s are some new constants depending on $C_{n,l,\rho^1,\rho^2,\dots,\rho^j}$'s. It is worth to note that the last term on the right hand side of (5.27) belongs to \mathcal{A}_{n+1} , so it remains to check whether the second term on right hand side of (5.27) also belongs to \mathcal{A}_{n+1} .

Differentiating the vorticity equation (4.3)₁ with respect to x ρ_1^1 times and y ρ_2^1 times, and then evaluating at $y=0$, we have, by using $u^\epsilon|_{y=0} = v^\epsilon|_{y=0} \equiv 0$ and denoting $e_2 := (0, 1)$,

$$(5.28) \quad \begin{aligned} &(\partial_t - \epsilon^2\partial_x^2) D^{\rho^1} \omega^\epsilon|_{y=0} \\ &= - \sum_{\substack{\beta \leq \rho^1 \\ \beta_2 \geq 1}} \binom{\rho^1}{\beta} D^{\beta - e_2} \omega^\epsilon \partial_x D^{\rho^1 - \beta} \omega^\epsilon + \sum_{\substack{\beta \leq \rho^1 \\ \beta_2 \geq 2}} \binom{\rho^1}{\beta} \partial_x D^{\beta - 2e_2} \omega^\epsilon \partial_y D^{\rho^1 - \beta} \omega^\epsilon + \partial_y^2 D^{\rho^1} \omega^\epsilon|_{y=0}. \end{aligned}$$

Using (5.28), one may justify by a routine counting of indices that the second term on the right hand side of (5.27) belongs to \mathcal{A}_{n+1} . This completes the proof of lemma 5.9. \square

5.1.2. *Weighted L^2 Estimate on g_s^ϵ .* In this subsection we will derive the L^2 estimate on $(1+y)^\gamma g_s^\epsilon$ by the standard energy method. This can be done because the quantity $g_s^\epsilon := \partial_x^s \omega^\epsilon - \frac{\partial_y \omega^\epsilon}{\omega^\epsilon} \partial_x^s (u^\epsilon - U)$ avoids the loss of x -derivative by a nonlinear cancellation, which is one of the key observations in this paper and will be explained as follows.

Let us begin by writing down the evolution equations for ω^ϵ and $u^\epsilon - U$:

$$(5.29) \quad \begin{cases} (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \omega^\epsilon = \epsilon^2 \partial_x^2 \omega^\epsilon + \partial_y^2 \omega^\epsilon \\ (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) (u^\epsilon - U) = \epsilon^2 \partial_x^2 (u^\epsilon - U) + \partial_y^2 (u^\epsilon - U) - (u^\epsilon - U) \partial_x U \end{cases}$$

where we applied the regularized Bernoulli's law (4.2) in the derivation of (5.29)₂. Since our aim is to control the H^s norm of ω^ϵ (or $u^\epsilon - U$), let us differentiate (5.29) with respect to x s times. Then we have

$$(5.30) \quad \begin{cases} (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \partial_x^s \omega^\epsilon + \partial_x^s v^\epsilon \partial_y \omega^\epsilon = \epsilon^2 \partial_x^{s+2} \omega^\epsilon + \partial_x^s \partial_y^2 \omega^\epsilon + \dots \\ (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \partial_x^s (u^\epsilon - U) + \partial_x^s v^\epsilon \omega^\epsilon = \epsilon^2 \partial_x^{s+2} (u^\epsilon - U) + \partial_x^s \partial_y^2 (u^\epsilon - U) + \dots, \end{cases}$$

where we applied the fact that $\omega^\epsilon = \partial_y (u^\epsilon - U)$ and the symbol \dots represents the lower order terms which we want the reader to ignore at this moment.

The main obstacle in (5.30) is the term $\partial_x^s v^\epsilon = -\partial_x^{s+1} \partial_y^{-1} u^\epsilon$, which has $s+1$ x -derivatives so that standard energy estimates cannot be closed. However, since there are two equations in (5.30), we can eliminate the problematic term $\partial_x^s v^\epsilon$ by subtracting them in an appropriate way.

Subtracting $\frac{\partial_y \omega^\epsilon}{\omega^\epsilon} \times (5.30)_2$ from $(5.30)_1$, we obtain

$$(5.31) \quad \begin{aligned} & (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) g_s^\epsilon \\ &= 2\epsilon^2 \left\{ \partial_x^{s+1} (u^\epsilon - U) - \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x^s (u^\epsilon - U) \right\} \partial_x a^\epsilon + 2g_s^\epsilon \partial_y a^\epsilon - g_1^\epsilon \partial_x^s U - \sum_{j=1}^{s-1} \binom{s}{j} g_{j+1}^\epsilon \partial_x^{s-j} u^\epsilon \\ & \quad - \sum_{j=1}^{s-1} \binom{s}{j} \partial_x^{s-j} v^\epsilon \{ \partial_x^j \partial_y \omega^\epsilon - a^\epsilon \partial_x^j \omega^\epsilon \} + a^\epsilon \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^j (u^\epsilon - U) \partial_x^{s-j+1} U. \end{aligned}$$

where $g_k^\epsilon := \partial_x^k \omega^\epsilon - a^\epsilon \partial_x^k (u^\epsilon - U)$ and $a^\epsilon := \frac{\partial_y \omega^\epsilon}{\omega^\epsilon}$. Here, the main reason that we can apply this nonlinear cancelation is the Oleinik's monotonicity assumption (i.e., $\omega > 0$), which is ensured in our solution class $H_{\sigma, \delta}^{s, \gamma}$. For the justification of (5.31), see appendix D.

Now, we are going to derive the following weighted energy estimate on g_s^ϵ :

Proposition 5.10 (L^2 Control on $(1+y)^\gamma g_s^\epsilon$). *Under the hypotheses of proposition 5.3, we have the following estimate:*

$$(5.32) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s^\epsilon\|_{L^2}^2 \\ & \leq -\frac{1}{2} \epsilon^2 \|(1+y)^\gamma \partial_x g_s^\epsilon\|_{L^2}^2 - \frac{1}{2} \|(1+y)^\gamma \partial_y g_s^\epsilon\|_{L^2}^2 + C \|\partial_x^{s+1} p^\epsilon\|_{L^2(\mathbb{T})}^2 + C_{\gamma,\delta} \|\partial_x^s U\|_{L^\infty(\mathbb{T})}^2 \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 \\ & \quad + C_{s,\gamma,\sigma,\delta} \{1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty(\mathbb{T})}\} \{\|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}, \end{aligned}$$

where the positive constants $C, C_{\gamma,\delta}$ and $C_{s,\gamma,\sigma,\delta}$ are independent of ϵ .

The proof of proposition 5.10 is almost a straight forward application of energy methods except the estimation (5.37) below is slightly tricky.

Proof of proposition 5.10. Multiplying the evolution equation (5.31) by $(1+y)^{2\gamma} g_s^\epsilon$, and then integrating over $\mathbb{T} \times \mathbb{R}^+$, we have

$$(5.33) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(1+y)^\gamma g_s^\epsilon\|_{L^2}^2 \\ & = \epsilon^2 \iint (1+y)^{2\gamma} g_s^\epsilon \partial_x^2 g_s^\epsilon + \iint (1+y)^{2\gamma} g_s^\epsilon \partial_y^2 g_s^\epsilon - \iint (1+y)^{2\gamma} g_s^\epsilon \{u^\epsilon \partial_x g_s^\epsilon + v^\epsilon \partial_y g_s^\epsilon\} \\ & \quad + 2\epsilon^2 \iint (1+y)^{2\gamma} g_s^\epsilon \{\partial_x^{s+1}(u^\epsilon - U) - \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x^s(u^\epsilon - U)\} \partial_x a^\epsilon \\ & \quad + 2 \iint (1+y)^{2\gamma} |g_s^\epsilon|^2 \partial_y a^\epsilon - \iint (1+y)^{2\gamma} g_1^\epsilon g_s^\epsilon \partial_x^s U \\ & \quad - \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_{j+1}^\epsilon g_s^\epsilon \partial_x^{s-j} u^\epsilon - \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} \partial_x^{s-j} v^\epsilon \{\partial_x^j \partial_y \omega^\epsilon - a^\epsilon \partial_x^j \omega^\epsilon\} g_s^\epsilon \\ & \quad + \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s^\epsilon a^\epsilon \partial_x^j (u^\epsilon - U) \partial_x^{s-j+1} U. \end{aligned}$$

Indeed, all terms on the right hand side of (5.33) can be controlled by using integration by parts and the standard Sobolev's type estimate on multilinear forms. Precisely, we have

Claim 5.11. *There exist constants $C, C_\delta, C_{\gamma,\delta}$ and $C_{s,\gamma,\sigma,\delta} > 0$ such that*

$$(5.34) \quad \epsilon^2 \iint (1+y)^{2\gamma} g_s^\epsilon \partial_x^2 g_s^\epsilon = -\epsilon^2 \|(1+y)^\gamma \partial_x g_s^\epsilon\|_{L^2}^2,$$

$$(5.35) \quad \iint (1+y)^{2\gamma} g_s^\epsilon \partial_y^2 g_s^\epsilon \leq -\frac{1}{2} \|(1+y)^\gamma \partial_y g_s^\epsilon\|_{L^2}^2 + C_{\gamma,\delta} \{1 + \|\partial_x^s U\|_{L^\infty(\mathbb{T})}^2\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 + C \|\partial_x^{s+1} p^\epsilon\|_{L^2(\mathbb{T})}^2,$$

$$(5.36) \quad \left| \iint (1+y)^{2\gamma} g_s^\epsilon \{u^\epsilon \partial_x g_s^\epsilon + v^\epsilon \partial_y g_s^\epsilon\} \right| \leq C_{s,\gamma,\sigma,\delta} \{\|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty}\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2,$$

$$\begin{aligned}
(5.37) \quad & \left| 2\epsilon^2 \iint (1+y)^{2\gamma} g_s^\epsilon \{ \partial_x^{s+1}(u^\epsilon - U) - \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x^s(u^\epsilon - U) \} \partial_x a^\epsilon \right| \\
& \leq \frac{1}{2} \epsilon^2 \| (1+y)^\gamma \partial_x g_s^\epsilon \|_{L^2}^2 + \epsilon^2 C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}},
\end{aligned}$$

$$(5.38) \quad \left| 2 \iint (1+y)^{2\gamma} |g_s^\epsilon|^2 \partial_y a^\epsilon \right| \leq C_\delta \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2,$$

$$(5.39) \quad \left| \iint (1+y)^{2\gamma} g_1^\epsilon g_s^\epsilon \partial_x^s U \right| \leq C_{s,\gamma,\sigma,\delta} \|\partial_x^s U\|_{L^\infty(\mathbb{T})} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}},$$

$$(5.40) \quad \left| \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_{j+1}^\epsilon g_s^\epsilon \partial_x^{s-j} u^\epsilon \right| \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \}^2 \|\omega^\epsilon\|_{H_g^{s,\gamma}},$$

$$\begin{aligned}
(5.41) \quad & \left| \sum_{j=1}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} \partial_x^{s-j} v^\epsilon \{ \partial_x^j \partial_y \omega^\epsilon - a^\epsilon \partial_x^j \omega^\epsilon \} g_s^\epsilon \right| \\
& \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2,
\end{aligned}$$

$$\begin{aligned}
(5.42) \quad & \left| \sum_{j=0}^{s-1} \binom{s}{j} \iint (1+y)^{2\gamma} g_s^\epsilon a^\epsilon \partial_x^j (u^\epsilon - U) \partial_x^{s-j+1} U \right| \\
& \leq C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}.
\end{aligned}$$

Assuming claim 5.11, which will be proven at the end of this subsection, for the moment, we can apply (5.34) - (5.42) to (5.33) to obtain our desired inequality (5.32) because $\epsilon \in [0, 1]$ and $\|\partial_x^s U\|_{L^2(\mathbb{T})} \leq \|\partial_x^s U\|_{L^\infty(\mathbb{T})} \leq \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} \leq \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}$. □

To complete the proof of proposition 5.10, we will show claim 5.11 as follows.

Proof of Claim 5.11.

Proof of (5.34):

The equality (5.34) follows directly from an integration by parts in the variable x .

Proof of (5.35):

Integrating by parts in the variable y (cf. remark 5.7), we have

$$\begin{aligned}
& \iint (1+y)^{2\gamma} g_s^\epsilon \partial_y^2 g_s^\epsilon \\
& = - \|(1+y)^\gamma \partial_y g_s^\epsilon\|_{L^2}^2 - \int_{\mathbb{T}} g_s^\epsilon \partial_y g_s^\epsilon dx|_{y=0} - 2\gamma \iint (1+y)^{2\gamma-1} g_s^\epsilon \partial_y g_s^\epsilon \\
(5.43) \quad & \leq - \frac{3}{4} \|(1+y)^\gamma \partial_y g_s^\epsilon\|_{L^2}^2 - \int_{\mathbb{T}} g_s^\epsilon \partial_y g_s^\epsilon dx|_{y=0} + C_\gamma \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.
\end{aligned}$$

In order to deal with the boundary integral $\int_{\mathbb{T}} g_s^\epsilon \partial_y g_s^\epsilon|_{y=0}$, one may apply the boundary conditions (4.1)₄ and (4.3)₃ to justify that

$$\partial_y g_s^\epsilon|_{y=0} = \partial_x^{s+1} p^\epsilon + \frac{\partial_y^2 \omega^\epsilon}{\omega^\epsilon} \partial_x^s U - a^\epsilon g_s^\epsilon|_{y=0}.$$

This boundary condition allows us to reduce the order of $\partial_y g_s^\epsilon$, and hence, using the simple trace estimate (5.15) and the facts that $\omega^\epsilon|_{y=0} \geq \delta$ and $\|a^\epsilon\|_{L^\infty} \leq \delta^{-2}$, one may prove that

$$(5.44) \quad \left| \int_{\mathbb{T}} g_s^\epsilon \partial_y g_s^\epsilon dx|_{y=0} \right| \leq \frac{1}{4} \|(1+y)^\gamma \partial_y g_s^\epsilon\|_{L^2}^2 + C_\delta \{1 + \|\partial_x^s U\|_{L^\infty}^2\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 + C \|\partial_x^{s+1} p^\epsilon\|_{L^2(\mathbb{T})}^2.$$

Substituting (5.44) into (5.43), we obtain (5.35).

Proof of (5.36):

Integrating by parts (cf. remark 5.7), and using $\partial_x u^\epsilon + \partial_y v^\epsilon = 0$, we have

$$(5.45) \quad \iint (1+y)^{2\gamma} g_s^\epsilon \{u^\epsilon \partial_x g_s^\epsilon + v^\epsilon \partial_y g_s^\epsilon\} = \gamma \iint (1+y)^{2\gamma-1} v^\epsilon |g_s^\epsilon|^2.$$

which and inequality (B.9) imply inequality (5.36).

Proof of (5.37):

Since $\omega^\epsilon \in C([0, T]; H_{\sigma, \delta}^{s+4, \gamma})$, it follows from the definition of $H_{\sigma, \delta}^{s+4, \gamma}$ that $(1+y)^\sigma \omega^\epsilon \geq \delta$ and $|(1+y)^{\sigma+\alpha_2} D^\alpha \omega^\epsilon| \leq \delta^{-1}$ for all $|\alpha| \leq 2$. Thus, we have $\|(1+y) \partial_x a^\epsilon\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}$ and $\left\| \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \right\|_{L^\infty} \leq \delta^{-2}$, and hence,

$$(5.46) \quad \left| 2\epsilon^2 \iint (1+y)^{2\gamma} g_s^\epsilon \left\{ \partial_x^{s+1} (u^\epsilon - U) - \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x^s (u^\epsilon - U) \right\} \partial_x a^\epsilon \right| \leq 2\epsilon^2 C_\delta \|(1+y)^\gamma g_s^\epsilon\|_{L^2} \{ \|(1+y)^{\gamma-1} \partial_x^{s+1} (u^\epsilon - U)\|_{L^2} + \|(1+y)^{\gamma-1} \partial_x^s (u^\epsilon - U)\|_{L^2} \}.$$

Now, we require the following inequality:

$$(5.47) \quad \|(1+y)^{\gamma-1} \partial_x^{s+1} (u^\epsilon - U)\|_{L^2} \leq C_{\gamma, \sigma, \delta} \{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma \partial_x g_s^\epsilon\|_{L^2} + \|(1+y)^{\gamma-1} \partial_x^s (u^\epsilon - U)\|_{L^2} \}.$$

Assuming (5.47) for the moment, we can apply it and proposition B.3 to (5.46), and obtain

$$\begin{aligned} & \left| 2\epsilon^2 \iint (1+y)^{2\gamma} g_s^\epsilon \left\{ \partial_x^{s+1} (u^\epsilon - U) - \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x^s (u^\epsilon - U) \right\} \partial_x a^\epsilon \right| \\ & \leq \epsilon^2 C_{s, \gamma, \sigma, \delta} \{ \|\omega^\epsilon\|_{H_g^{s, \gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} + \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma \partial_x g_s^\epsilon\|_{L^2} \} \|\omega^\epsilon\|_{H_g^{s, \gamma}} \end{aligned}$$

which implies (5.37) by Cauchy's inequality and the inequality $\|\partial_x^s U\|_{L^2(\mathbb{T})} \leq \frac{1}{2\pi} \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})}$.

To complete the justification of (5.37), we have to verify (5.47) as follows.

Since $\delta \leq (1+y)^\sigma \omega^\epsilon \leq \delta^{-1}$ and $u^\epsilon|_{y=0} = 0$, applying part (ii) of lemma B.1, we have

$$(5.48) \quad \begin{aligned} \|(1+y)^{\gamma-1} \partial_x^{s+1}(u^\epsilon - U)\|_{L^2} &\leq \delta^{-1} \left\| (1+y)^{\gamma-\sigma-1} \frac{\partial_x^{s+1}(u^\epsilon - U)}{\omega^\epsilon} \right\|_{L^2} \\ &\leq C_{\gamma,\sigma,\delta} \left\{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \left\| (1+y)^\gamma \omega^\epsilon \partial_y \left(\frac{\partial_x^{s+1}(u^\epsilon - U)}{\omega^\epsilon} \right) \right\|_{L^2} \right\}. \end{aligned}$$

It is worth to note that

$$\omega^\epsilon \partial_y \left(\frac{\partial_x^{s+1}(u^\epsilon - U)}{\omega^\epsilon} \right) = g_{s+1}^\epsilon = \partial_x g_s^\epsilon + \partial_x a^\epsilon \partial_x^s (u^\epsilon - U),$$

so by (5.48), we have

$$\begin{aligned} &\|(1+y)^{\gamma-1} \partial_x^{s+1}(u^\epsilon - U)\|_{L^2} \\ &\leq C_{\gamma,\sigma,\delta} \{ \|\partial_x^{s+1} U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma \partial_x g_s^\epsilon\|_{L^2} + \|(1+y) \partial_x a^\epsilon\|_{L^\infty} \|(1+y)^{\gamma-1} \partial_x^s (u^\epsilon - U)\|_{L^2} \} \end{aligned}$$

which implies inequality (5.47) because $\|(1+y) \partial_x a^\epsilon\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}$.

Proof of (5.38):

The inequality (5.38) follows from $\|\partial_y a^\epsilon\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}$ and the definition of $\|\omega^\epsilon\|_{H_g^{s,\gamma}}$.

Proof of (5.39) and (5.40):

Both inequalities (5.39) and (5.40) follows from the Hölder inequality and proposition B.3.

Proof of (5.41):

For $j = 2, 3, \dots, s-1$, using proposition B.3 and $\|(1+y)a^\epsilon\|_{L^\infty} \leq \delta^{-2}$, we have

$$(5.49) \quad \begin{aligned} &\left| \iint (1+y)^{2\gamma} \partial_x^{s-j} v^\epsilon \{ \partial_x^j \partial_y \omega^\epsilon - a^\epsilon \partial_x^j \omega^\epsilon \} g_s^\epsilon \right| \\ &\leq \left\| \frac{\partial_x^{s-j} v^\epsilon}{1+y} \right\|_{L^\infty} \{ \|(1+y)^{\gamma+1} \partial_x^j \partial_y \omega^\epsilon\|_{L^2} + \|(1+y)a^\epsilon\|_{L^\infty} \|(1+y)^\gamma \partial_x^j \omega^\epsilon\|_{L^2} \} \|(1+y)^\gamma g_s^\epsilon\|_{L^2} \\ &\leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2. \end{aligned}$$

When $j = 1$, using proposition B.3 and $\|(1+y)a^\epsilon\|_{L^\infty} \leq \delta^{-2}$ again, we have

$$\begin{aligned} &\left| \iint (1+y)^{2\gamma} \partial_x^{s-1} v^\epsilon \{ \partial_x \partial_y \omega^\epsilon - a^\epsilon \partial_x \omega^\epsilon \} g_s^\epsilon \right| \\ &\leq \left\{ \left\| \frac{\partial_x^{s-1} v^\epsilon + y \partial_x^s U}{1+y} \right\|_{L^2} \left(\|(1+y)^{\gamma+1} \partial_x \partial_y \omega^\epsilon\|_{L^\infty} + \|(1+y)a^\epsilon\|_{L^\infty} \|(1+y)^\gamma \partial_x \omega^\epsilon\|_{L^\infty} \right) \right. \\ &\quad \left. + \|\partial_x^s U\|_{L^\infty(\mathbb{T})} \left(\|(1+y)^{\gamma+1} \partial_x \partial_y \omega^\epsilon\|_{L^2} + \|(1+y)a^\epsilon\|_{L^\infty} \|(1+y)^\gamma \partial_x \omega^\epsilon\|_{L^2} \right) \right\} \|(1+y)^\gamma g_s^\epsilon\|_{L^2} \end{aligned}$$

$$(5.50) \\ \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.$$

Combining estimates (5.49) and (5.50), we prove inequality (5.41).

Proof of (5.42):

For any $j = 0, 1, \dots, s-1$, by proposition B.3 and $\|(1+y)a^\epsilon\|_{L^\infty} \leq \delta^{-2}$, we have

$$\begin{aligned} & \left| \iint (1+y)^{2\gamma} g_s^\epsilon a^\epsilon \partial_x^j (u^\epsilon - U) \partial_x^{s-j+1} U \right| \\ & \leq \|(1+y)^\gamma g_s^\epsilon\|_{L^2} \|(1+y)a^\epsilon\|_{L^\infty} \|(1+y)^{\gamma-1} \partial_x^j (u^\epsilon - U)\|_{L^2} \|\partial_x^{s-j+1} U\|_{L^\infty(\mathbb{T})} \\ & \leq C_{s,\gamma,\sigma,\delta} \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}}, \end{aligned}$$

which implies inequality (5.42). □

5.1.3. Weighted H^s Estimate on ω^ϵ . The aim of this subsubsection is to combine the estimates in proposition 5.6 and proposition 5.10 to derive the growth rate control (5.1) on the weighted H^s energy of ω^ϵ .

According to propositions 5.6 and 5.10, we know from the definition of $\|\cdot\|_{H_g^{s,\gamma}}$ that

$$\begin{aligned} & \frac{d}{dt} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 \\ & \leq C_{s,\gamma,\sigma,\delta} \{ 1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^\infty(\mathbb{T})} \} \{ \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})} \} \|\omega^\epsilon\|_{H_g^{s,\gamma}} \\ & \quad + C_{\gamma,\delta} \|\partial_x^s U\|_{L^\infty(\mathbb{T})}^2 \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 + C_{s,\gamma,\sigma,\delta} \{ 1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}} \}^{s-2} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 + C_s \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l \partial_x p^\epsilon\|_{H^{s-2l}(\mathbb{T})}^2 \\ & \leq C_{s,\gamma,\sigma,\delta} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^s + C_{s,\gamma,\sigma,\delta} \{ 1 + \|\partial_x^{s+1} U\|_{L^\infty(\mathbb{T})}^4 \} + C_s \sum_{l=0}^{\frac{s}{2}} \|\partial_t^l \partial_x p^\epsilon\|_{H^{s-2l}(\mathbb{T})}^2, \end{aligned}$$

and hence, it follows from the comparison principle of ordinary differential equations that

$$\begin{aligned} & \|\omega^\epsilon(t)\|_{H_g^{s,\gamma}}^2 \\ & \leq \left\{ \|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right\} \left\{ 1 - \left(\frac{s}{2} - 1 \right) C_{s,\gamma,\sigma,\delta} \left(\|\omega_0\|_{H_g^{s,\gamma}}^2 + \int_0^t F(\tau) d\tau \right)^{\frac{s-2}{2}} t \right\}^{-\frac{2}{s-2}} \end{aligned}$$

as long as the second braces on the right hand side of the above inequality is positive, where $F : [0, T] \rightarrow \mathbb{R}^+$ is defined by (5.2). This proves inequality (5.1).

5.2. Weighted L^∞ Estimates on Lower Order Terms. In this subsection we will derive uniform (in ϵ) weighted L^∞ estimates on $D^\alpha \omega^\epsilon$ for $|\alpha| \leq 2$ by using the classical maximum principle. The key idea is to “view” the evolution equation of $D^\alpha \omega^\epsilon$ as a “linear”

parabolic equation with coefficients involving higher order terms of u^ϵ, v^ϵ and ω^ϵ , which can be controlled by proposition B.3 provided that $\|\omega^\epsilon\|_{H_g^{s,\gamma}} < +\infty$.

More precisely, we will prove part (ii) of proposition 5.3 as follows.

Proof of part (ii) of proposition 5.3. This proof, based on a simple application of the classical maximum principle for parabolic equations, will be divided into two steps. In the first step, we will derive weighted L^∞ controls on $D^\alpha \omega^\epsilon$ by using the maximum principle stated in appendix E. These controls will rely on the boundary values of $D^\alpha \omega^\epsilon$ at $y = 0$, so we will also derive estimates on the boundary values of $D^\alpha \omega^\epsilon$ by using Sobolev embedding or growth rate control argument in the second step.

Step 1: (Maximum Principle Argument)

First of all, let us derive a L^∞ estimate on the $I := \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega^\epsilon|^2$ as follows.

For notational convenience, let us denote, for $|\alpha| \leq 2$,

$$B_\alpha := (1+y)^{\sigma+\alpha_2} D^\alpha \omega^\epsilon$$

which is our concerned quantities. By a direct computation, B_α satisfies

$$(5.51) \quad \{\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2\} B_\alpha = Q_\alpha \partial_y B_\alpha + R_\alpha B_\alpha + S_\alpha,$$

where the quantities Q_α, R_α and S_α are given explicitly by

$$Q_\alpha := -\frac{2(\sigma + \alpha_2)}{1+y}, \quad R_\alpha := \frac{\sigma + \alpha_2}{1+y} v^\epsilon + \frac{(\sigma + \alpha_2)(\sigma + \alpha_2 + 1)}{(1+y)^2},$$

$$S_\alpha := \begin{cases} 0 & \text{if } \alpha = (0, 0), \\ -\sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \{(1+y)^{\beta_2} D^\beta u^\epsilon B_{\alpha-\beta+e_1} + (1+y)^{\beta_2-1} D^\beta v^\epsilon B_{\alpha-\beta+e_2}\} & \text{if } |\alpha| \geq 1. \end{cases}$$

Here, $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Using proposition B.3, we have the following pointwise controls on Q_α, R_α and S_α : for $|\alpha| \leq 2$,

$$(5.52) \quad \begin{cases} |Q_\alpha| \leq C_\sigma, & |R_\alpha| \leq C_{s,\gamma,\sigma,\delta} \{1 + \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\}, \\ |S_\alpha| \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\} \sum_{0 < \beta \leq \alpha} \{|B_{\alpha-\beta+e_1}| + |B_{\alpha-\beta+e_2}|\}, \end{cases}$$

where C_σ and $C_{s,\gamma,\sigma,\delta}$ are some universal constants which are independent of the solution ω^ϵ .

Let us recall from the definition that $I := \sum_{|\alpha| \leq 2} |B_\alpha|^2$, so using (5.51) and (5.52), we have

$$\begin{aligned} & \{\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2\} I \\ &= -2 \sum_{|\alpha| \leq 2} \{\epsilon^2 |\partial_x B_\alpha|^2 + |\partial_y B_\alpha|^2\} + 2 \sum_{|\alpha| \leq 2} \{Q_\alpha B_\alpha \partial_y B_\alpha + R_\alpha |B_\alpha|^2 + S_\alpha B_\alpha\} \end{aligned}$$

$$\leq C_{s,\gamma,\sigma,\delta}\{1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}I.$$

Applying the classical maximum principle for parabolic equations (see lemma E.1 for instance) to the quantity I , we have, after using the definition (5.4) of G ,

$$(5.53) \quad \begin{aligned} & \|I(t)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \\ & \leq \max\{e^{C_{s,\gamma,\sigma,\delta}\{1+G(t)\}t} \|I(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0,t]} \{e^{C_{s,\gamma,\sigma,\delta}\{1+G(t)\}(t-\tau)} \|I(\tau)\|_{y=0}\|_{L^\infty(\mathbb{T})}\}\}. \end{aligned}$$

Next, we are going to derive a lower bound estimate on $B_{(0,0)} := (1+y)^\sigma \omega^\epsilon$. To do this, let us recall that $B_{(0,0)}$ satisfies

$$\{\partial_t + u^\epsilon \partial_x + (v^\epsilon - Q_{(0,0)}) \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2\} B_{(0,0)} = R_{(0,0)} B_{(0,0)}.$$

Using the classical maximum principle (see lemma E.2 for instance) and (5.52), we obtain

$$(5.54) \quad \begin{aligned} & \min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega^\epsilon(t) \\ & \geq (1 - C_{s,\gamma,\sigma,\delta}\{1+G(t)\}t e^{C_{s,\gamma,\sigma,\delta}\{1+G(t)\}t}) \min\{\min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega_0, \min_{[0,t] \times \mathbb{T}} \omega^\epsilon|_{y=0}\}. \end{aligned}$$

Step 2: (Controls on Boundary Values)

According to inequalities (5.53) and (5.54), we have already controlled the underlying quantities I and $B_{(0,0)} := (1+y)^\sigma \omega^\epsilon$ by their initial and boundary values. However, their boundary values are not given in the problem, so we will estimate them in this step.

In order to control $I|_{y=0} := \sum_{|\alpha| \leq 2} |D^\alpha \omega^\epsilon|_{y=0}|^2$, we will apply Sobolev embedding argument and growth rate control argument in the cases $s \geq 4$ and $s \geq 6$ respectively. Combining the boundary estimates on I with (5.53), we will finally obtain inequalities (5.3) and (5.5).

To derive inequality (5.3), we first apply lemma B.2 to obtain

$$\begin{aligned} \|I|_{y=0}\|_{L^\infty(\mathbb{T})} & \leq \sum_{|\alpha| \leq 2} \|I\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq 3C^2 \sum_{|\alpha| \leq 2} \{\|D^\alpha \omega^\epsilon\|_{L^2}^2 + \|\partial_x D^\alpha \omega^\epsilon\|_{L^2}^2 + \|\partial_y^2 D^\alpha \omega^\epsilon\|_{L^2}^2\} \\ & \leq 6C^2 \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2 \end{aligned}$$

which and inequality (5.53) imply inequality (5.3).

To derive inequality (5.5), let us begin by writing down the evolution equation for $D^\alpha \omega^\epsilon$ at $y = 0$: for any $|\alpha| \leq 2$,

$$(5.55) \quad \partial_t D^\alpha \omega^\epsilon \Big|_{y=0} = (\epsilon^2 \partial_x^2 + \partial_y^2) D^\alpha \omega^\epsilon + E_\alpha \Big|_{y=0}$$

where the term E_α is given explicitly by

$$E_\alpha := \begin{cases} 0 & \text{if } \alpha_2 = 0 \\ -\omega^\epsilon \partial_x \partial_y^{\alpha_2-1} \omega^\epsilon & \text{if } \alpha_1 = 0 \text{ and } \alpha_2 \geq 1 \\ -|\partial_x \omega^\epsilon|^2 - \omega^\epsilon \partial_x^2 \omega^\epsilon & \text{if } \alpha = (1, 1). \end{cases}$$

Here, the derivation of (5.55) is just a direct differentiation on the vorticity equation (4.3)₁ as well as using the boundary condition $u^\epsilon|_{y=0} = v^\epsilon|_{y=0} \equiv 0$. Furthermore, by proposition B.3, if $s \geq 4$, then

$$(5.56) \quad \|E_\alpha|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2.$$

In addition, if $s \geq |\alpha| + 4$, then by proposition B.3 again, for $\epsilon \in [0, 1]$,

$$(5.57) \quad \|(\epsilon^2 \partial_x^2 + \partial_y^2) D^\alpha \omega^\epsilon|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \|\omega^\epsilon\|_{H_g^{s,\gamma}}.$$

Therefore, using (5.55) - (5.57) and inequality (B.10), we have, for $s \geq 6$,

$$\|\partial_t I|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \{1 + \|\omega^\epsilon\|_{H_g^{s,\gamma}}\} \|\omega^\epsilon\|_{H_g^{s,\gamma}}^2,$$

and hence, by direction integration and definition (5.4) of Ω , we obtain

$$(5.58) \quad \|I(t)|_{y=0}\|_{L^\infty(\mathbb{T})} \leq \|I(0)|_{y=0}\|_{L^\infty(\mathbb{T})} + C_{s,\gamma} \{1 + \Omega(t)\} \Omega(t)^2 t.$$

Combining estimates (5.53) and (5.58), we prove our desired estimate (5.5).

Lastly, it remains to show inequality (5.6). Let us begin by deriving an estimate on $\omega^\epsilon|_{y=0}$.

Using identity (5.55), inequality (5.57) and definition (5.4) of Ω , we have, for any $s \geq 4$,

$$\|\partial_t \omega^\epsilon|_{y=0}\|_{L^\infty(\mathbb{T})} \leq C_{s,\gamma} \Omega(t),$$

so a direct integration yields

$$(5.59) \quad \min_{\mathbb{T}} \omega^\epsilon(t)|_{y=0} \geq \min_{\mathbb{T}} \omega_0|_{y=0} - C_{s,\gamma} \Omega(t) t.$$

Combining estimates (5.54) and (5.59), we prove (5.6). □

6. PROOF OF THE MAIN THEOREM

The purpose of this section is to complete the proof of our main theorem 2.2. In other words, we will prove existence and uniqueness to the Prandtl equations (2.1) in the subsection 6.1 and 6.2 respectively.

6.1. Existence for the Prandtl Equations. In this subsection we will construct the solution to the Prandtl equations (2.1) by passing to the limit ϵ goes to 0^+ in the regularized Prandtl equations (4.1). Our proof will be based on the uniform (in ϵ) weighted estimates derived in proposition 5.3. Using these estimates, we will derive uniform bounds and lifespan on ω^ϵ , and then prove convergence of ω^ϵ and consistency of the limit ω as follows.

Uniform Bounds and Life-span on ω^ϵ

According to proposition 5.1, a solution ω^ϵ to the regularized Prandtl equations (4.1) exists up to a time interval $[0, T_{s,\gamma,\sigma,\delta,\epsilon,\omega_0,U}]$, which may depend on ϵ as well. However, in

proposition 5.3 we have already derived uniform (in ϵ) estimates on ω^ϵ , so one may apply the standard continuous induction argument to further solve the Prandtl equations (2.1) up to a time interval which is independent of ϵ . As a result, we have

Proposition 6.1 (Uniform Life-span and Estimates for ω^ϵ). *In addition to the hypotheses of proposition 5.1, when $s = 4$, we further assume that $\delta > 0$ is chosen small enough such that the initial hypothesis (2.7) holds. Then there exists a uniform life-span $T := T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}}, U) > 0$, which is independent of ϵ , such that the regularized vorticity system (4.3) - (4.4) has a solution $\omega^\epsilon \in C([0, T]; H_{\sigma,\delta}^{s,\gamma}) \cap C^1([0, T]; H^{s-2,\gamma})$ with the following uniform (in ϵ) estimates:*

(i) (Uniform Weighted H^s Estimate) For any $\epsilon \in [0, 1]$ and any $t \in [0, T]$,

$$(6.1) \quad \|\omega^\epsilon(t)\|_{H_g^{s,\gamma}} \leq 4\|\omega_0\|_{H_g^{s,\gamma}}.$$

(ii) (Uniform Weighted L^∞ Bound) For any $\epsilon \in [0, 1]$ and $t \in [0, T]$,

$$(6.2) \quad \left\| \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega^\epsilon(t)| \right\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}^2 \leq \frac{1}{\delta^2}.$$

(iii) (Uniform Weighted L^∞ Lower Bound) For any $\epsilon \in [0, 1]$ and $t \in [0, T]$,

$$(6.3) \quad \min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega^\epsilon(t) \geq \delta.$$

Proof. The uniform life-span $T := T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s,\gamma}}, U)$ can be guaranteed by the uniform estimates (6.1) - (6.3), so it suffices to justify them. Indeed, the life-span T can be taken as $\min\{T_1, T_2, T_3\}$ where T_1, T_2 and T_3 will be defined below.

(i) According to the definition (5.2) of F and the regularized Bernoulli's law (4.2),

$$\|F\|_{L^\infty} \leq C_{s,\gamma,\sigma,\delta} \left\{ 1 + \sum_{l=0}^{\frac{s}{2}+1} \|\partial_t^l U\|_{H^{s-2l+2}(\mathbb{T})}^2 \right\}^2 \leq C_{s,\gamma,\sigma,\delta} M_U$$

where $M_U := \sup_t \left\{ 1 + \sum_{l=0}^{\frac{s}{2}+1} \|\partial_t^l U\|_{H^{s-2l+2}(\mathbb{T})}^2 \right\}^2 < +\infty$, so if we take $T_1 := \min \left\{ \frac{3\|\omega_0\|_{H_g^{s,\gamma}}^2}{C_{s,\gamma,\sigma,\delta} M_U}, \frac{1 - 2^{-s+2}}{2^{s-2} C_{s,\gamma,\sigma,\delta} \|\omega_0\|_{H_g^{s,\gamma}}^{s-2}} \right\}$, then by inequality (5.1), estimate (6.1) holds for all $t \in [0, T_1]$.

(ii) When $s \geq 6$, using part (i) of proposition 6.1, we know from the definition (5.4) of Ω and G that for any $t \in [0, T_1]$ where T_1 is defined in part (i),

$$(6.4) \quad \Omega(t) \leq 4\|\omega_0\|_{H_g^{s,\gamma}} \quad \text{and} \quad G(t) \leq 4\|\omega_0\|_{H_g^{s,\gamma}} + M_U =: K.$$

Thus, if we take $T_2 := \min \left\{ T_1, \frac{1}{64\delta^2 C_{s,\gamma}(1+4\|\omega_0\|_{H_g^{s,\gamma}})\|\omega_0\|_{H_g^{s,\gamma}}^2}, \frac{\ln 2}{C_{s,\gamma,\sigma,\delta}(1+K)} \right\}$, then using inequality (5.5) and the initial assumption $\sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega_0|^2 \leq \frac{1}{4\delta^2}$, we have the upper bound (6.2) for all $t \in [0, T_2]$.

When $s = 4$, using inequality (5.3), estimate (6.4) and the initial hypothesis (2.7), we also have the upper bound (6.2) for all $t \in [0, T_2]$.

(iii) Let us take $T_3 := \min \left\{ T_1, \frac{\delta}{8C_{s,\gamma}\|\omega_0\|_{H_g^{s,\gamma}}}, \frac{1}{6C_{s,\gamma,\sigma,\delta}(1+K)}, \frac{\ln 2}{C_{s,\gamma,\sigma,\delta}(1+K)} \right\}$. Then using inequalities (5.6) and (6.4), we know that the lower bound (6.3) holds for all $t \in [0, T_3]$. \square

Convergence and Consistency

Using almost equivalence relation (A.1) and uniform weighted H^s estimate (6.1), we have

$$(6.5) \quad \sup_{0 \leq t \leq T} (\|\omega^\epsilon\|_{H^{s,\gamma}} + \|u^\epsilon - U\|_{H^{s,\gamma-1}}) \leq C_{s,\gamma,\sigma,\delta} \{4\|\omega_0\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\} < +\infty.$$

Furthermore, using evolution equations (5.29), uniform H^s bound (6.5) and proposition B.3, one also find that $\partial_t \omega^\epsilon$ and $\partial_t(u^\epsilon - U)$ are uniformly (in ϵ) bounded in $L^\infty([0, T]; H^{s-2,\gamma})$ and $L^\infty([0, T]; H^{s-2,\gamma-1})$ respectively. By the Lions-Aubin Lemma and the compact embedding of $H^{s,\gamma}$ in $H_{loc}^{s'}$ stated in lemma 6.2, we have, taking a subsequence if necessary, as $\epsilon_k \rightarrow 0^+$,

$$(6.6) \quad \begin{cases} \omega^{\epsilon_k} \xrightarrow{*} \omega \text{ in } L^\infty([0, T]; H^{s,\gamma}) & \text{and} & \omega^{\epsilon_k} \rightarrow \omega \text{ in } C([0, T]; H_{loc}^{s'}), \\ u^{\epsilon_k} - U \xrightarrow{*} u - U \text{ in } L^\infty([0, T]; H^{s,\gamma-1}) & \text{and} & u^{\epsilon_k} \rightarrow u \text{ in } C([0, T]; H_{loc}^{s'}), \end{cases}$$

for all $s' < s$, where $\omega \in L^\infty([0, T]; H^{s,\gamma}) \cap \bigcap_{s' < s} C([0, T]; H_{loc}^{s'})$, $u - U \in L^\infty([0, T]; H^{s,\gamma-1}) \cap \bigcap_{s' < s} C([0, T]; H_{loc}^{s'})$ and $\omega = \partial_y u$. Using the local uniform convergence of $\partial_x u^{\epsilon_k}$, we also have

the pointwise convergence of v^{ϵ_k} : as $\epsilon_k \rightarrow 0^+$,

$$(6.7) \quad v^{\epsilon_k} = - \int_0^y \partial_x u^{\epsilon_k} dy \rightarrow - \int_0^y \partial_x u dy =: v.$$

Combining (6.6) - (6.7), one may justify the pointwise convergences of all terms in the regularized Prandtl equations (4.1)₁ - (4.1)₄. Thus, passing to the limit $\epsilon_k \rightarrow 0^+$ in (4.1)₁ - (4.1)₄ and the regularized Bernoulli's law (4.2), we know that the limit (u, v) solves the Prandtl equations (2.1)₁ - (2.1)₄ with the Bernoulli's law (2.3) in the classical sense.

Lastly, in order to complete the proof of consistency, it remains to justify that $\omega \in L^\infty([0, T]; H_{\sigma,\delta}^{s,\gamma})$ and the matching condition (2.1)₅. Since $D^\alpha \omega^{\epsilon_k}$ converges to $D^\alpha \omega$

pointwisely for all $|\alpha| \leq 2$ and that ω^{ϵ_k} satisfies

$$(6.8) \quad \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega^{\epsilon_k}|^2 \leq \frac{1}{\delta^2} \quad \text{and} \quad (1+y)^\sigma \omega^{\epsilon_k} \geq \delta,$$

we deduce that (6.8) still holds for ω , and hence, $\omega \in L^\infty(0, T; H_{\sigma, \delta}^{s, \gamma})$.

Also, by the Lebesgue's dominated convergence theorem,

$$\int_0^{+\infty} \omega \, dy = \lim_{\epsilon_k \rightarrow 0^+} \int_0^{+\infty} \omega^{\epsilon_k} \, dy = U$$

which is equivalent to the matching condition (2.1)₅ because $\omega = \partial_y u > 0$.

To complete the proof of existence, let us state and prove the following

Lemma 6.2. *Let s be a positive integer, $\gamma' \geq 0$ and $M < +\infty$. Assume*

$$(6.9) \quad \|f^\epsilon\|_{H^{s, \gamma'}} \leq M$$

for all $\epsilon \in (0, 1]$. Then there exist a function $f \in H^{s, \gamma'}$ and a sequence $\{\epsilon_k\}_{k \in \mathbb{N}} \subseteq (0, 1]$ with $\lim_{k \rightarrow +\infty} \epsilon_k = 0^+$ such that as $\epsilon_k \rightarrow 0^+$,

$$(6.10) \quad f^{\epsilon_k} \xrightarrow{H^{s, \gamma'}} f \quad \text{and} \quad f^{\epsilon_k} \xrightarrow{H_{loc}^{s'}} f \quad \text{for all } s' < s.$$

Proof of lemma 6.2. First of all, let us mention that $H^{s, \gamma'}$ has an inner product structure:

$$\langle \phi, \psi \rangle_{H^{s, \gamma'}} := \sum_{|\alpha| \leq s} \int_0^{+\infty} \int_{\mathbb{T}} (1+y)^{2\gamma'+2\alpha_2} D^\alpha \phi D^\alpha \psi,$$

so the uniform bound (6.9) implies the weak convergence of f^{ϵ_k} in (6.10) via the Banach-Alaoglu theorem.

Next, by the definition of $\|\cdot\|_{H^{s, \gamma'}}$, $\|f^\epsilon\|_{H^s} \leq \|f^\epsilon\|_{H^{s, \gamma'}} \leq M$. This implies the local $H^{s'}$ norm convergence in (6.10) for all $s' < s$ because of the standard compactness of $H^s(\mathbb{T} \times \mathbb{R}^+)$. \square

Finally, let us end this subsection by giving the following

Remark 6.3 (Life-span for $U \equiv \text{constant}$). In the special case that $U > 0$ is a constant, one may prove that the life-span T stated in our main theorem 2.2 is independent of the constant value of U , that is, $T := T(s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s, \gamma}})$. The reasoning is as follows:

When $U \equiv \text{constant}$, it follows from the regularized Bernoulli's law (4.2) that $\partial_x p^\epsilon \equiv 0$, so by definitions (5.2) and (5.4), we have $F \equiv C_{s, \gamma, \sigma, \delta}$ and $G(t) = \Omega(t) = \sup_{[0, t]} \|\omega^\epsilon\|_{H_g^{s, \gamma}}$ where all of F , G and Ω are independent of U . As a result, all of our weighted estimates (5.1), (5.3), (5.5) and (5.6) are independent of U . Therefore, one may slightly modify the proof of proposition 6.1 to show that uniform weighted estimates (6.1) - (6.3) hold in a time interval $[0, T_{s, \gamma, \sigma, \delta, \|\omega_0\|_{H^{s, \gamma}}}]$, which is independent of U . According to our proof of convergence and

consistency in subsection 6.1, we can solve the solution (u, v) of the Prandtl equations (2.1) in the same time interval, and hence, the life-span T stated in the main theorem 2.2 is also independent of U .

6.2. Uniqueness for the Prandtl Equations. The aim of this section is to prove the uniqueness of $H_{\sigma,\delta}^{s,\gamma}$ solutions constructed in subsection 6.1. To show the uniqueness, we will generalize the nonlinear cancelation applied in subsubsection 5.1.2 to the L^2 comparison of two $H_{\sigma,\delta}^{s,\gamma}$ solutions. This motivates us to consider the quantity \tilde{g} below.

Specifically, the uniqueness of $H_{\sigma,\delta}^{s,\gamma}$ solutions to the Prandtl equations (2.1) is a direct consequence of the following L^2 comparison principle.

Proposition 6.4 (L^2 Comparison Principle). *For any $s \geq 4, \gamma \geq 1, \sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$, let (u_i, v_i) solve the Prandtl equations (2.1) with the vorticity $\omega_i := \partial_y u_i \in C([0, T]; H_{\sigma,\delta}^{s,\gamma}) \cap C^1([0, T]; H^{s-2,\gamma})$ for $i = 1, 2$. Define $\tilde{g} := \omega_1 - \omega_2 + \frac{\partial_y \omega_2}{\omega_2}(u_1 - u_2)$. Then we have*

$$(6.11) \quad \|\tilde{g}(t)\|_{L^2}^2 + \int_0^t \|\partial_y \tilde{g}\|_{L^2}^2 \leq \|\tilde{g}(0)\|_{L^2}^2 + C_{\gamma,\sigma,\delta,\omega,U} \int_0^t \|\tilde{g}\|_{L^2}^2$$

where the constant $C_{\gamma,\sigma,\delta,\omega,U}$ depends on $\gamma, \sigma, \delta, \|\omega_1\|_{H_g^{4,\gamma}}, \|\omega_2\|_{H_g^{4,\gamma}}$ and $\|\partial_x^4 U\|_{L^2(\mathbb{T})}$ only.

Applying the Gronwall's lemma to (6.11), we obtain

$$\|\tilde{g}(t)\|_{L^2}^2 \leq \|\tilde{g}(0)\|_{L^2}^2 e^{C_{\gamma,\sigma,\delta,\omega,U} t}$$

which implies $\tilde{g} \equiv 0$ provided that $u_1|_{t=0} = u_2|_{t=0}$. Since $\omega_2 \partial_y \left(\frac{u_1 - u_2}{\omega_2} \right) = \tilde{g} \equiv 0$, we have

$$(6.12) \quad u_1 - u_2 = q \omega_2$$

for some function $q := q(t, x)$. Using the Oleinik's monotonicity assumption $\omega_2 > 0$ and the Dirichlet boundary condition $u_i|_{y=0} \equiv 0$ for $i = 1, 2$, we know via (6.12) that $q \equiv 0$, and hence, $u_1 \equiv u_2$. Since v_i can be uniquely determined by u_i , we also have $v_1 \equiv v_2$. This proves the uniqueness of $H_{\sigma,\delta}^{s,\gamma}$ solutions.

In the rest of this subsection, we will prove proposition 6.4 as follows.

Proof of proposition 6.4. Let us denote $(\tilde{u}, \tilde{v}) = (u_1, v_1) - (u_2, v_2)$ and $a_2 := \frac{\partial_y \omega_2}{\omega_2}$. Then one may check that $\tilde{g} = \tilde{\omega} - a_2 \tilde{u} = \omega_2 \partial_y \left(\frac{\tilde{u}}{\omega_2} \right)$ and satisfies

$$(6.13) \quad (\partial_t + u_1 \partial_x + v_1 \partial_y - \partial_y^2) \tilde{g} = -2\tilde{\omega} \partial_y a_2 - \tilde{u} \{ \tilde{u} \partial_x a_2 + \tilde{v} \partial_y a_2 + 2a_2 \partial_y a_2 \}.$$

To derive the L^2 estimates on \tilde{g} , let us first recall that we define the cutoff function $\chi_R(y) := \chi(\frac{y}{R})$ for any $R \geq 1$, where $\chi \in C_c^\infty([0, +\infty))$ satisfies the properties (4.7). Then

χ_R has the following pointwise properties: as $R \rightarrow +\infty$,

$$\chi_R \rightarrow \mathbf{1}_{\mathbb{R}^+}, \quad |\chi'_R| \leq \frac{2}{R} \rightarrow 0^+ \quad \text{and} \quad |\chi''_R| \leq O\left(\frac{1}{R^2}\right) \rightarrow 0^+.$$

For any $t \in (0, T]$, multiplying equation (6.13) by $2\chi_R \tilde{g}$, and then integrating over $[0, t] \times \mathbb{T} \times \mathbb{R}^+$, we obtain, via integration by parts,

$$\begin{aligned} & \iint \chi_R \tilde{g}^2(t) dy dx - \iint \chi_R \tilde{g}^2|_{t=0} dy dx \\ &= -2 \int_0^t \iint \chi_R |\partial_y \tilde{g}|^2 - 2 \int_0^t \int_{\mathbb{T}} \tilde{g} \partial_y \tilde{g}|_{y=0} dx - 4 \int_0^t \iint \chi_R \tilde{g} \tilde{\omega} \partial_y a_2 \\ & \quad - 2 \int_0^t \iint \chi_R \tilde{g} \tilde{u} \{ \tilde{u} \partial_x a_2 + \tilde{v} \partial_y a_2 + 2a_2 \partial_y a_2 \} + \mathcal{R}_1 + \mathcal{R}_2 \\ &\leq -2 \int_0^t \iint \chi_R |\partial_y \tilde{g}|^2 - 2 \int_0^t \int_{\mathbb{T}} \tilde{g} \partial_y \tilde{g}|_{y=0} dx + 4 \|\partial_y a_2\|_{L^\infty} \int_0^t \|\tilde{g}\|_{L^2} \|\tilde{\omega}\|_{L^2} \\ (6.14) \quad & + 2 \|(1+y) \{ \tilde{u} \partial_x a_2 + \tilde{v} \partial_y a_2 + 2a_2 \partial_y a_2 \}\|_{L^\infty} \int_0^t \|\tilde{g}\|_{L^2} \left\| \frac{\tilde{u}}{1+y} \right\|_{L^2} + \mathcal{R}_1 + \mathcal{R}_2, \end{aligned}$$

where the remainder terms \mathcal{R}_i are defined by

$$\mathcal{R}_1 := \int_0^t \iint \chi'_R v_1 \tilde{g}^2 \quad \text{and} \quad \mathcal{R}_2 := \int_0^t \iint \chi''_R \tilde{g}^2.$$

Now, the first technical problem is to deal with the boundary integral $\int_0^t \int_{\mathbb{T}} \tilde{g} \partial_y \tilde{g}|_{y=0} dx$. Since $\partial_y \tilde{g}|_{y=0} = -a_2 \tilde{g}|_{y=0}$, we have, after applying the simple trace estimate (5.15),

$$(6.15) \quad \left| \int_0^t \int_{\mathbb{T}} \tilde{g} \partial_y \tilde{g}|_{y=0} dx \right| \leq \frac{1}{2} \int_0^t \iint \chi_R |\partial_y \tilde{g}|^2 + C \{ \|a_2\|_{L^\infty} + \|a_2\|_{L^\infty}^2 + \|\partial_y a_2\|_{L^\infty} \} \int_0^t \iint \tilde{g}^2.$$

Furthermore, since $\omega_2 \in H_{\sigma, \delta}^{s, \gamma}$, it follows from the weighted L^∞ bounds on ω_2 that

$$(6.16) \quad \begin{cases} \|(1+y)a_2\|_{L^\infty} \leq \delta^{-2} \\ \|(1+y)\partial_x a_2\|_{L^\infty}, \|(1+y)^2 \partial_y a_2\|_{L^\infty} \leq \delta^{-2} + \delta^{-4}, \end{cases}$$

so by proposition B.3,

$$(6.17) \quad \|(1+y) \{ \tilde{u} \partial_x a_2 + \tilde{v} \partial_y a_2 + 2a_2 \partial_y a_2 \}\|_{L^\infty} \leq C_{\gamma, \sigma, \delta} \{ 1 + \|\omega_1\|_{H_g^{4, \gamma}} + \|\omega_2\|_{H_g^{4, \gamma}} + \|\partial_x^4 U\|_{L^2(\mathbb{T})} \}.$$

Substituting (6.15) - (6.17) into (6.14), we obtain

$$\begin{aligned} & \iint \chi_R \tilde{g}^2(t) dy dx - \iint \chi_R \tilde{g}^2|_{t=0} dy dx \\ (6.18) \quad & \leq - \int_0^t \iint \chi_R |\partial_y \tilde{g}|^2 + C_\delta \int_0^t \|\tilde{g}\|_{L^2}^2 \\ & \quad + C_{\gamma, \sigma, \delta, \|\omega_1\|_{H_g^{4, \gamma}}, \|\omega_2\|_{H_g^{4, \gamma}}, \|\partial_x^4 U\|_{L^2}} \int_0^t \|\tilde{g}\|_{L^2} \left\{ \|\tilde{\omega}\|_{L^2} + \left\| \frac{\tilde{u}}{1+y} \right\|_{L^2} \right\} + \mathcal{R}_1 + \mathcal{R}_2. \end{aligned}$$

Next, we emphasize that both $\tilde{\omega}$ and $\frac{\tilde{u}}{1+y}$ can be controlled by \tilde{g} , namely,

Claim 6.5.

$$(6.19) \quad \|\tilde{\omega}\|_{L^2}, \left\| \frac{\tilde{u}}{1+y} \right\|_{L^2} \leq C_{\sigma,\delta} \|\tilde{g}\|_{L^2}.$$

The proof of claim 6.5 is very similar to that of lemma A.2, so we will only outline it at the end of this subsection. Assuming claim 6.5 for the moment, we can substitute (6.19) into (6.18) to obtain

$$(6.20) \quad \begin{aligned} & \iint \chi_R \tilde{g}^2(t) dy dx - \iint \chi_R \tilde{g}^2|_{t=0} dy dx \\ & \leq - \int_0^t \iint \chi_R |\partial_y \tilde{g}|^2 + C_{\gamma,\sigma,\delta, \|\omega_1\|_{H_g^{4,\gamma}}, \|\omega_2\|_{H_g^{4,\gamma}}, \|\partial_x^4 U\|_{L^2}} \int_0^t \|\tilde{g}\|_{L^2}^2 + \mathcal{R}_1 + \mathcal{R}_2. \end{aligned}$$

Finally, both integrands of \mathcal{R}_1 and \mathcal{R}_2 can be controlled by a multiple of \tilde{g}^2 , which belongs to $L^1([0, T]; \mathbb{T} \times \mathbb{R}^+)$, so applying Lebesgue's dominated convergence theorem, we have

$$(6.21) \quad \lim_{R \rightarrow +\infty} \mathcal{R}_i = 0 \quad \text{for } i = 1, 2.$$

Using monotone convergence theorem and (6.21), we can pass to the limit $R \rightarrow +\infty$ in (6.20) to obtain (6.11).

Lastly, we will justify claim 6.5 as follows.

Proof of claim 6.5. Using triangle inequality and (6.16)₁, we have $\|\tilde{\omega}\|_{L^2} \leq \|\tilde{g}\|_{L^2} + \delta^{-2} \left\| \frac{\tilde{u}}{1+y} \right\|_{L^2}$, so it suffices to control \tilde{u} .

Since $\delta \leq (1+y)^\sigma \omega_2 \leq \delta^{-1}$ and $\tilde{u}|_{y=0} \equiv 0$, applying part (ii) of lemma B.1, we obtain

$$\left\| \frac{\tilde{u}}{1+y} \right\|_{L^2} \leq \delta^{-1} \left\| (1+y)^{-\sigma-1} \frac{\tilde{u}}{\omega_2} \right\|_{L^2} \leq C_{\sigma,\delta} \left\| (1+y)^{-\sigma} \partial_y \left(\frac{\tilde{u}}{\omega_2} \right) \right\|_{L^2} \leq C_{\sigma,\delta} \|\tilde{g}\|_{L^2}$$

because $\tilde{g} = \omega_2 \partial_y \left(\frac{\tilde{u}}{\omega_2} \right)$. □

□

7. EXISTENCE FOR THE REGULARIZED PRANDTL EQUATIONS

The aim of this section is to solve the regularized Prandtl equations (4.1), or equivalently its vorticity system (4.3) - (4.4). In other words, we will prove proposition 5.1 according to the plan described in section 4. However, we will only sketch our proof because the method for solving intermediate approximate systems (4.3) - (4.4), (4.5) - (4.6) and (4.8) - (4.9) is standard. Before we proceed, it should be also remarked that we will solve the approximate systems (4.8) - (4.9), (4.5) - (4.6) and (4.3) - (4.4) with a decreasing order of regularities. The main reason of this technical arrangement is to derive our estimates in a rigorous way so

that we can differentiate the intermediate equations pointwisely and have enough pointwise decay at $y = +\infty$ according to prop C.1.

7.1. Solvability of Inhomogenous Heat Equation. In this first subsection we will solve an inhomogenous heat equation in the weighted space $H_{\sigma,\delta}^{s,\gamma}$. This existence result will be applied to solve the linearized, truncated and regularized vorticity system (4.8) - (4.9) in the next subsection.

Let us consider the following inhomogenous heat equation: for any $\epsilon > 0$,

$$(7.1) \quad \begin{cases} \partial_t W + F_R = \epsilon^2 \partial_x^2 W + \partial_y^2 W & \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+ \\ W|_{t=0} = W_0 & \text{on } \mathbb{T} \times \mathbb{R}^+ \\ \partial_y W|_{y=0} = \partial_x p^\epsilon & \text{on } [0, T] \times \mathbb{T} \end{cases}$$

where W is an unknown, W_0 and $\partial_x p^\epsilon$ are given initial and boundary data, F_R is a given inhomogenous term with compact support in $[0, T] \times \mathbb{T} \times [0, 2R]$. Since (7.1) is just a standard inhomogenous heat equation, we can solve it by classical methods and obtain the following

Proposition 7.1 (Existence of Inhomogenous Heat Equation). *Let $s \geq 4$ be an even integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$, $\delta \in (0, \frac{1}{2})$ and $\epsilon \in (0, 1]$. If $W_0 \in H_{\sigma,2\delta}^{s+12,\gamma}$ and $\text{supp } F_R \subseteq [0, T] \times \mathbb{T} \times [0, 2R]$, then there exist a time $T := T(s, \gamma, \sigma, \delta, R, \|W_0\|_{H_{\sigma,\delta}^{s+8,\gamma}}, \|F_R\|_{C^2}) > 0$ and a solution $W \in C([0, T]; H_{\sigma,\delta}^{s+8,\gamma}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ to the inhomogenous heat equation (7.1).*

Furthermore, we have the following pointwise decay at $y = +\infty$: for any $l = 0, 1, \dots, \frac{s}{2} + 4$, for any $|\alpha| \leq s - 2l + 9$,

$$(7.2) \quad \partial_t^l D^\alpha W = \begin{cases} O((1+y)^{-\sigma-\alpha_2}) & \text{if } |\alpha| + 2l \leq 2 \\ O((1+y)^{-\frac{\sigma+(2|\alpha|+2l-2-1)\gamma}{2|\alpha|+2l-2}-\alpha_2}) & \text{if } 2 \leq |\alpha| + 2l \leq s + 9, \end{cases}$$

and energy estimate:

$$(7.3) \quad \begin{aligned} & \frac{d}{dt} \| \| W \| \|_{s+8,\gamma}^2 + \epsilon^2 \| \| \partial_x W \| \|_{s+8,\gamma}^2 + \| \| \partial_y W \| \|_{s+8,\gamma}^2 \\ & \leq C_{s,\gamma} \| \| W \| \|_{s+8,\gamma}^2 + C_s \| \| W \| \|_{s+8,\gamma} \| F_R \|_{s+8,\gamma} + C_s \| \| F_R \| \|_{s+7,\gamma}^2 + C_s \| \| \partial_x p^\epsilon \| \|_{s+8}^2, \end{aligned}$$

where the norms $\| \| \cdot \| \|_{s',\gamma}$ and $\| \| \cdot \| \|_{s'}$ are defined in definition B.4.

Outline of the Proof. Using the method of reflection and Duhamel's principle, one may express the unique global-in-time $C([0, T] \times \mathbb{T} \times \mathbb{R}^+) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ solution to (7.1) by an explicit solution formula

$$(7.4) \quad W = K_{W_0} + K_{\partial_x p^\epsilon} + K_{F_R}$$

where the terms K_{W_0} , $K_{\partial_x p^\epsilon}$ and K_{F_R} can be written explicitly by using the Gaussian (i.e., the heat kernel), and depend on W_0 , $\partial_x p^\epsilon$ and F_R respectively. Since the solution formula

(7.4) is explicit, based on the properties of the Gaussian, one may prove the following two facts: as $y \rightarrow +\infty$,

- (i) both $K_{\partial_x p^\epsilon}$ and K_{F_R} decay exponentially fast;
- (ii) the term $D^\alpha K_{W_0} \lesssim$ (or \gtrsim) $(1+y)^{-b_\alpha}$ provided that $D^\alpha W_0$ does.

Using quantitative versions of facts (i) and (ii), one can justify by (7.4) that W fulfills all weighted L^∞ controls for $H_{\sigma,\delta}^{s+8,\gamma}$ within a short time interval $[0, T_{s,\gamma,\sigma,\delta,R,\|W_0\|_{H^{s,\gamma}},\|F_R\|_{C^2}}]$.

Furthermore, applying proposition C.1 with $s' = s + 8$, we know that W_0 satisfies the pointwise decay (7.2) for $l = 0$ and $|\alpha| \leq s + 9$, and hence, W does. Using the heat equation (7.1)₁ repeatedly, we also obtain the pointwise decay (7.2) in our desired ranges of l and α .

Finally, it remains to show the energy estimate (7.3), but its proof just follows from standard energy methods, so we will omit the proof here. However, during the estimation, one requires to apply an integration by parts in the y -direction to deal with the operator ∂_y^2 , so we would like to give the following two remarks on the boundary values of W :

- (I) (Boundary Values at $y = 0$) The boundary values of W as well as its derivatives can be reconstructed by using the boundary reduction formula

$$\partial_y^{2k+1} W|_{y=0} = (\partial_t - \epsilon^2 \partial_x^2)^k \partial_x p^\epsilon + \sum_{j=0}^{k-1} (\partial_t - \epsilon^2 \partial_x^2)^{k-j-1} \partial_y^{2j+1} F_R|_{y=0},$$

which reduces the order of the boundary terms so that we can control the boundary integral at $y = 0$ via the simple trace estimate (5.15);

- (II) (Boundary Values at $y = +\infty$) All boundary terms of W as well as its derivatives required for deriving energy estimate (7.3) actually vanish fast enough at $y = +\infty$ because of the pointwise decay estimate (7.2). Thus, all required boundary integrals at $y = +\infty$ are zero.

□

7.2. Solvability of Linearized, Truncated and Regularized Vorticity System. Based on the $H_{\sigma,\delta}^{s,\gamma}$ solutions to the inhomogeneous heat equation (7.1) derived in subsection 7.1, we will construct a sequence of solutions to the linearized, truncated and regularized vorticity system (4.8) - (4.9) with uniform bounds in this subsection. This sequence of solutions as well as their uniform bounds will be the foundation for solving the truncated and regularized vorticity system (4.5) - (4.6) in the next subsection.

Let us begin by defining an iterative sequence $\{(u^n, v^n, \omega^n)\}_{n \in \mathbb{N}}$ as follows:

- (i) $\omega^0(t, x, y) := \omega_0(x, y)$;
- (ii) (u^n, v^n) is defined by formulae (4.9) for all $n \in \mathbb{N}$;
- (iii) ω^{n+1} is defined to be the $C([0, T]; H_{\sigma,\delta}^{s+8,\gamma}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ solution to the linearized, truncated and regularized vorticity system (4.8) - (4.9) for all $n \in \mathbb{N}$.

The natural question is whether the iterative sequence $\{(u^n, v^n, \omega^n)\}_{n \in \mathbb{N}}$ is well-defined, and the answer is affirmative because of the following

Proposition 7.2 (Existence of Linearized, Truncated and Regularized Vorticity System).

Let $s \geq 4$ be an even integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, \frac{1}{2}), \epsilon \in (0, 1]$ and $R \geq 1$. If $\omega_0 \in H_{\sigma, 2\delta}^{s+12, \gamma}$ and $\sup_t |||U|||_{s+9, \infty} < +\infty$ where $||| \cdot |||_{s', \infty}$ is defined in definition B.4, then there exist a uniform life-span $T := T(s, \gamma, \sigma, \delta, \epsilon, \chi, R, \|\omega_0\|_{H^{s+8, \gamma}}, \sup_t |||U|||_{s+9, \infty}) > 0$ which is independent of n , and a sequence of solutions $\{\omega^n\}_{n \in \mathbb{N}} \subseteq C([0, T]; H_{\sigma, \delta}^{s+8, \gamma}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$ to the linearized, truncated and regularized vorticity system (4.8) - (4.9).

Furthermore, the pointwise decay estimate (7.2) holds for $W := \omega^n$ for all $n \in \mathbb{N}$ and we have the following uniform (in n) energy estimate: for all $n \in \mathbb{N}$ and for all $t \in [0, T]$,

$$(7.5) \quad |||\omega^n|||_{s+8, \gamma}^2 + \epsilon^2 \int_0^t |||\partial_x \omega^n|||_{s+8, \gamma}^2 + \int_0^t |||\partial_y \omega^n|||_{s+8, \gamma}^2 \leq \mathcal{Q}_{s+10}(\|\omega_0\|_{H^{s+8, \gamma}})$$

where the norm $||| \cdot |||_{s', \gamma}$ is defined in definition B.4 and \mathcal{Q}_l is a degree l polynomial with non-negative coefficients which depends on $C_{s, \gamma, \chi}$ and $|||U|||_{s+8, \infty}$ only.

Outline of the proof. For a given $\omega^n \in C([0, T]; H_{\sigma, \delta}^{s+8, \gamma}) \cap C^\infty((0, T] \times \mathbb{T} \times \mathbb{R}^+)$, the local in time solvability of ω^{n+1} in the same function space follows directly by applying proposition 7.1 with $W := \omega^{n+1}$ and $F_R := \chi_R \{u^n \partial_x \omega^n + v^n \partial_y \omega^n\}$, although the life-span T may depend on n a priori. However, the uniform (in n) energy estimate (7.5) guarantee the uniform (in n) life-span T by the standard continuous induction argument, so it suffices to prove (7.5).

In order to derive the energy estimate (7.5), we have to control $|||F_R|||_{s+7, \gamma}$ and $|||F_R|||_{s+8, \gamma}$ for $F_R := \chi_R \{u^n \partial_x \omega^n + v^n \partial_y \omega^n\}$. Using the triangle inequality, proposition B.5 and proposition B.6, one may check that

$$(7.6) \quad \begin{cases} |||F_R|||_{s+7, \gamma} \leq C_{s, \gamma, \chi} |||\omega^n|||_{s+8, \gamma}^2 + |||U|||_{s+8, \infty}^2 \\ |||F_R|||_{s+8, \gamma} \leq C_{s, \gamma, \chi, \mathbb{R}} \{ |||\omega^n|||_{s+8, \gamma} + |||U|||_{s+9, \infty} \} \cdot \{ |||\partial_x \omega^n|||_{s+8, \gamma} + |||\partial_y \omega^n|||_{s+8, \gamma} \} \end{cases}$$

where the norms $||| \cdot |||_{s', \infty, \gamma}$ and $||| \cdot |||_{s', \infty}$ are defined in definition B.4.

Applying inequalities (7.3) and (7.6), one can easily show that as long as $|||\omega^n|||_{s+8, \gamma}|_{t=0} \leq L$, there exists a uniform (in n) time interval $[0, T_{s, \gamma, \sigma, \delta, \epsilon, \chi, R, \sup_t |||U|||_{s+9, \infty}, L}]$ such that

$$(7.7) \quad |||\omega^n|||_{s+8, \gamma}^2 + \epsilon^2 \int_0^t |||\partial_x \omega^n|||_{s+8, \gamma}^2 + \int_0^t |||\partial_y \omega^n|||_{s+8, \gamma}^2 \leq 4L^2 \quad \text{for all } n \in \mathbb{N}$$

because both (7.3) and (7.6) are independent of n . Therefore, it remains to derive a uniform (in n) control on the initial data $|||\omega^n|||_{s+8, \gamma}|_{t=0}$.

To estimate $|||\omega^n|||_{s+8, \gamma}|_{t=0}$, let us first state without proof the following fact:

$$(7.8) \quad \|\partial_t^l \omega^n\|_{H^{s-2l+8, \gamma}}|_{t=0} \leq \mathcal{P}_{l+1}(\|\omega_0\|_{H^{s+8, \gamma}})$$

where \mathcal{P}_{l+1} is a degree $l+1$ polynomial defined by

$$\mathcal{P}_1(Z) := Z \quad \text{and} \quad \mathcal{P}_{l+1} := \mathcal{P}_l + C_{s,\gamma,\chi} \sum_{j=0}^{l-1} (\mathcal{P}_{j+1} + \|U\|_{s+8,\infty}) \mathcal{P}_{l-j} \quad \text{for all } l \geq 1.$$

The fact (7.8) can be proved by induction on (n, l) together with the following estimate:

$$(7.9) \quad Y_{n+1,l+1} \leq Y_{n+1,l} + C_{s,\gamma,\chi} \sum_{j=0}^l (Y_{n,j} + \|U\|_{s+8,\infty}) Y_{n,l-j}$$

where $Y_{n,l} := \|\partial_t^l \omega^n\|_{H^{s-2l+8,\gamma}}|_{t=0}$. The derivation of (7.9), which is based on (4.8)₁, proposition B.5 and proposition B.6, will be left for the reader.

Combining estimates (7.7) and (7.8), we show the uniform energy estimate (7.5) for

$$\mathcal{Q}_{s+10} := 4 \sum_{l=0}^{\frac{s}{2}+4} \mathcal{P}_{l+1}^2.$$

□

7.3. Solvability of Truncated and Regularized Vorticity System. The aim of this subsection is to construct a solution to the truncated and regularized vorticity system (4.5) - (4.6) by passing to the limit in its linearized version (4.8) - (4.9), which was solved with uniform bounds in subsection 7.2.

In other words, we will prove the following

Proposition 7.3 (Existence of Truncated and Regularized Vorticity System). *Let $s \geq 4$ be an even integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}, \delta \in (0, \frac{1}{2}), \epsilon \in (0, 1]$ and $R \geq 1$. If $\omega_0 \in H_{\sigma, 2\delta}^{s+12,\gamma}$ and $\sup_t \|U\|_{s+9,\infty} < +\infty$ where $\|\cdot\|_{s',\infty}$ is defined in definition B.4, then there exist a time $T := T(s, \gamma, \sigma, \delta, \epsilon, \chi, R, \|\omega_0\|_{H^{s+8,\gamma}}, \sup_t \|U\|_{s+9,\infty}) > 0$ and a solution $\omega_R \in C([0, T]; H_{\sigma,\delta}^{s+8,\gamma}) \cap$*

$\bigcap_{l=1}^{\frac{s}{2}+4} C^l([0, T]; H^{s-2l+8,\gamma})$ to the truncated and regularized vorticity system (4.5) - (4.6).

Furthermore, we have the following uniform (in R) weighted energy estimate:

$$(7.10) \quad \begin{aligned} & \frac{d}{dt} \|\omega_R\|_{s+4,\gamma}^2 + \epsilon^2 \|\partial_x \omega_R\|_{s+4,\gamma}^2 + \|\partial_y \omega_R\|_{s+4,\gamma}^2 \\ & \leq C_{s,\gamma,\epsilon,\chi} \{1 + \|\omega_R\|_{s+4,\gamma} + \|U\|_{s+6}\}^2 \|\omega_R\|_{s+4,\gamma}^2 + C_s \{1 + \|U\|_{s+6}\}^2 \|U\|_{s+6}^2 \end{aligned}$$

and the following weighted L^∞ estimates:

$$(7.11) \quad \|I_R(t)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq (\|I_R(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} + C_{s,\gamma} \Lambda(t) \sup_{[0,t]} \|\omega_R\|_{H^{s+4,\gamma}} t) e^{C_\sigma \Lambda(t)t}$$

$$(7.12) \quad \min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega_R(t) \geq (1 - \Lambda(t)t e^{\Lambda(t)t}) \left(\min_{\mathbb{T} \times \mathbb{R}^+} (1+y)^\sigma \omega_0 - C_\sigma \Lambda(t) \sup_{[0,t]} \|\omega_R\|_{H^{s+4,\gamma}} t \right),$$

where the norms $||| \cdot |||_{s', \gamma}$ and $||| \cdot |||_{s'}$ are defined in definition B.4, and the quantities I_R and Λ are defined by

$$I_R(t) := \sum_{|\alpha| \leq 2} |(1+y)^{\sigma+\alpha_2} D^\alpha \omega_R(t)|^2 \quad \text{and} \quad \Lambda(t) := 1 + \sup_{[0,t]} \|\omega_R\|_{H^{s+4,\gamma}} + \sup_{[0,t]} \|U\|_{C^3(\mathbb{T})}.$$

Outline of the proof. According to proposition 7.2, the sequence of solutions $\{\omega^n\}_{n \in \mathbb{N}}$ to the linearized, truncated and regularized vorticity system (4.8) - (4.9) has a uniform (in n) life-span $[0, T_{s,\gamma,\sigma,\delta,\epsilon,\chi,R,\omega_0,U}]$, in which $|||\omega^n|||_{s+8,\gamma}$ is uniformly bounded by estimate (7.5). Based on this uniform bound, one may apply the standard energy methods to $\omega^{n+1} - \omega^n$ to prove that the approximate sequence $\{\omega^n\}_{n \in \mathbb{N}}$ is indeed Cauchy in the norm $\sup_{t \in [0,T]} ||| \cdot |||_{s+6,\gamma}$ where the time $T := T(s, \gamma, \sigma, \delta, \epsilon, \chi, R, \|\omega_0\|_{s+8,\gamma}, \sup_t \|U\|_{s+9,\infty}) > 0$ is independent of n . As a result, we can pass to the limit n goes to $+\infty$ in (4.8) - (4.9) to obtain a solution $\omega_R := \lim_{n \rightarrow +\infty} \omega^n$ to the truncated and regularized vorticity system (4.5) - (4.6). Moreover,

ω_R belongs to $C([0, T]; H_{\sigma,\delta}^{s+8,\gamma}) \cap \bigcap_{l=1}^{\frac{s}{2}+4} C^l([0, T]; H^{s-2l+8,\gamma})$ because ω^n does.

The uniform energy estimate (7.10) follows from the standard energy methods, so its proof will be omitted here. It is noteworthy to mention that unlike the estimates in subsection 7.2, all constants in (7.10) are independent of R . This improvement is based on applying integration by parts appropriately to the integral involving the convection term $\chi_R \{u_R \partial_x \omega_R + v_R \partial_y \omega_R\}$, but it does not exist in (4.8) - (4.9) because the linearization destroys this structure.

The weighted L^∞ controls (7.11) and (7.12) can be derived by the classical maximum principle (see lemmas E.1 and E.2 for instance) as in subsection 5.2. We leave this for the interested reader.

□

7.4. Solvability of Regularized Vorticity System and Regularized Prandtl Equations. In this subsection we will construct a solution ω^ϵ to the regularized vorticity system (4.3) - (4.4) by passing to the limit in its truncated version (4.5) - (4.6), whose local-in-time solvability and uniform bounds are shown in subsection 7.3. Furthermore, we will also justify that ω^ϵ solves the regularized Prandtl equations (4.1).

More precisely, we will complete the proof of proposition 5.1 as follows.

Outline of the proof of proposition 5.1. To solve the regularized vorticity system (4.3) - (4.4), we first pick any function χ with the properties (4.7) in the truncated and regularized vorticity system (4.5) - (4.6). Then by proposition 7.3, we have a local-in-time solution ω_R to (4.5) - (4.6) and uniform bounds (7.10) - (7.12) on ω_R . Since the estimates (7.10) - (7.12) are independent of R , one can show that there exists a uniform time $T := T(s, \gamma, \sigma, \delta, \epsilon, \|\omega_0\|_{H^{s+4,\gamma}}, U) > 0$ such that $\{\omega_R\}_{R \geq 1} \subseteq C([0, T]; H_{\sigma,\delta}^{s+4,\gamma}) \cap C^1([0, T]; H^{s+2,\gamma})$

and $\|\omega_R\|_{H^{s+4,\gamma}} \leq C_{s,\gamma,\sigma,\delta,\epsilon,\|\omega_0\|_{H^{s+4,\gamma}},U}$ for all $R \geq 1$. Therefore, by the standard compactness argument, there exist a function $\omega^\epsilon \in C([0, T]; H_{\sigma,\delta}^{s+4,\gamma}) \cap C^1([0, T]; H^{s+2,\gamma})$ and a subsequence $\{R_k\}_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow +\infty} R_k = +\infty$ such that ω_{R_k} converges to ω^ϵ in $C([0, T]; H_{loc}^{s+2})$ as $R_k \rightarrow +\infty$. As a result, we can pass to the limit $R_k \rightarrow +\infty$ in (4.5) - (4.6), and prove that ω^ϵ solves the regularized vorticity system (4.3) - (4.4) in a classical sense.

Finally, we will justify that (u^ϵ, v^ϵ) defined by (4.4) satisfies the regularized Prandtl equations (4.1) as follows.

First of all, the matching condition (4.1)₅, the Dirichlet boundary condition $v^\epsilon|_{y=0}$ and the initial condition (4.1)₃ follows immediately from the formulae (4.4), $\omega_0 := \partial_y u_0$ and the compatibility condition (2.2). Then by direct differentiations on (4.4), we also have the incompressibility condition (4.1)₂ and $\omega^\epsilon = \partial_y u^\epsilon$.

To justify equation (4.1)₁, we substitute $\omega^\epsilon = \partial_y u^\epsilon$ into (4.3)₁ and obtain, via using (4.1)₂,

$$(7.13) \quad \partial_y \{ \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \omega^\epsilon \} = \partial_y \{ \epsilon^2 \partial_x^2 u^\epsilon + \partial_y \omega^\epsilon \}.$$

Then one may derive (4.1)₁ by integrating (7.13) with respect to y over $[y, +\infty)$ and using the following pointwise convergences: as $y \rightarrow +\infty$,

$$(7.14) \quad \begin{cases} v^\epsilon \omega^\epsilon, \partial_y \omega^\epsilon \rightarrow 0 \\ \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon \rightarrow \partial_t U + U \partial_x U - \epsilon^2 \partial_x^2 U =: -\partial_x p^\epsilon. \end{cases}$$

The pointwise convergences (7.14) can be shown easily as long as $\omega^\epsilon \in C([0, T]; H_{\sigma,\delta}^{s+4,\gamma}) \cap C^1([0, T]; H^{s+2,\gamma})$, so we leave this to the interested reader.

Lastly, it remains to show the Dirichlet boundary condition $u^\epsilon|_{y=0} = 0$. To prove this, we evaluate the evolution equation (4.1)₁ at $y = 0$, and apply the boundary conditions $v^\epsilon|_{y=0} = 0$ and (4.3)₃ to obtain that $u^\epsilon|_{y=0}$ satisfies the viscous Burger's equation:

$$(7.15) \quad \partial_t (u^\epsilon|_{y=0}) + (u^\epsilon|_{y=0}) \partial_x (u^\epsilon|_{y=0}) = \epsilon^2 \partial_x^2 (u^\epsilon|_{y=0}).$$

It follows from the classical uniqueness result for the viscous Burger's equation (7.15) that $u^\epsilon|_{y=0} \equiv 0$ since it does initially according to the compatibility condition (2.2). □

APPENDIX A. ALMOST EQUIVALENCE OF WEIGHTED NORMS

The purpose of this appendix is to justify the almost equivalence relation (3.2). In other words, we will prove the following

Proposition A.1 (Almost Equivalence of Weighted H^s Norms). *Let $s \geq 4$ be an integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$. For any $\omega \in H_{\sigma,\delta}^{s,\gamma}(\mathbb{T} \times \mathbb{R}^+)$, we have the following inequality: there exist constants c_δ and $C_{s,\gamma,\sigma,\delta} > 0$ such that*

$$(A.1) \quad c_\delta \|\omega\|_{H_g^{s,\gamma}} \leq \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \leq C_{s,\gamma,\sigma,\delta} \{ \|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2} \}$$

provided that $\omega = \partial_y u$, $u|_{y=0} = 0$ and $\lim_{y \rightarrow +\infty} u = U$, where the weighted H^s norms $\|\cdot\|_{H^{s,\gamma}}$ and $\|\cdot\|_{H_g^{s,\gamma}}$ are defined by (2.5) and (3.1) respectively.

Proof. Without loss of generality, we only need to prove inequality (A.1) for the smooth ω because the full version can be recovered by the standard density argument. First of all, it follows from the definition of $\|\omega\|_{H^{s,\gamma}}$ and $\|u - U\|_{H^{s,\gamma-1}}$ that

$$(A.2) \quad \begin{aligned} \|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} &\leq \|\omega\|_{H^{s,\gamma}} + \|u - U\|_{H^{s,\gamma-1}} \\ &\leq 2\{\|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2}\}. \end{aligned}$$

Furthermore, applying Wirtinger's inequality in the variable x repeatedly and part (i) of lemma B.1, we have

$$\begin{cases} \sum_{k=1}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \leq \frac{1 + \pi^{-2s}}{1 - \pi^{-2}} \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} \\ \|(1+y)^{\gamma-1} (u - U)\|_{L^2} \leq \frac{2}{2\gamma - 1} \|(1+y)^\gamma \omega\|_{L^2} \leq \frac{2}{2\gamma - 1} \|\omega\|_{H^{s,\gamma}}, \end{cases}$$

and hence, there exists a constant $C_{s,\gamma} > 0$ such that

$$(A.3) \quad \begin{aligned} \|\omega\|_{H^{s,\gamma}} + \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} &\leq \|\omega\|_{H^{s,\gamma}} + \sum_{k=0}^s \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \\ &\leq C_{s,\gamma} \{\|\omega\|_{H^{s,\gamma}} + \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2}\}. \end{aligned}$$

Therefore, according to inequalities (A.2) and (A.3), it suffices to prove

$$(A.4) \quad c_\delta \|\omega\|_{H_g^{s,\gamma}} \leq \|\omega\|_{H^{s,\gamma}} + \|(1+y)^{\gamma-1} \partial_x^s (u - U)\|_{L^2} \leq C_{\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}$$

for some constants c_δ and $C_{\gamma,\sigma,\delta} > 0$.

The key idea of proving (A.4) is the following

Lemma A.2 (L^2 Comparison of $\partial_x^k (u - U)$, $\partial_x^k \omega$ and g_k). *Let $s \geq 4$ be an integer, $\gamma \geq 1$, $\sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$. If $\omega \in H_{\sigma,\delta}^{s,\gamma}(\mathbb{T} \times \mathbb{R}^+)$, then for any $k = 1, 2, \dots, s$,*

$$(A.5) \quad \|(1+y)^\gamma g_k\|_{L^2} \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2}$$

where $g_k := \partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k (u - U)$. In addition, if $u|_{y=0} = 0$, then for any $k = 1, 2, \dots, s$,

$$(A.6) \quad \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \|(1+y)^{\gamma-1} \partial_x^k (u - U)\|_{L^2} \leq C_{\gamma,\sigma,\delta} \{ \|(1+y)^\gamma g_k\|_{L^2} + \|\partial_x^k U\|_{L^2(\mathbb{T})} \}$$

where $C_{\gamma,\sigma,\delta}$ is a constant depending on γ, σ and δ only.

Assuming lemma A.2, which will be shown at the end of this appendix, for the moment, we can show inequality (A.4) as follows.

Applying lemma A.2 for $k = s$, we obtain, from (A.5) and (A.6),

$$(A.7) \quad \begin{aligned} \frac{1}{2}\delta^4\|(1+y)^\gamma g_s\|_{L^2}^2 &\leq \|(1+y)^\gamma \partial_x^s \omega\|_{L^2}^2 + \|(1+y)^{\gamma-1} \partial_x^s(u-U)\|_{L^2}^2 \\ &\leq C_{\gamma,\sigma,\delta}\{\|(1+y)^\gamma g_s\|_{L^2}^2 + \|\partial_x^s U\|_{L^2(\mathbb{T})}^2\}. \end{aligned}$$

Adding $\sum_{\substack{|\alpha|\leq s \\ \alpha_1\leq s-1}} \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2}^2$ to (A.7), we have

$$\frac{1}{2}\delta^4\|\omega\|_{H_g^{s,\gamma}}^2 \leq \|\omega\|_{H^{s,\gamma}}^2 + \|(1+y)^{\gamma-1} \partial_x^s(u-U)\|_{L^2}^2 \leq C_{\gamma,\sigma,\delta}\{\|\omega\|_{H_g^{s,\gamma}}^2 + \|\partial_x^s U\|_{L^2(\mathbb{T})}^2\}$$

which implies inequality (A.4). □

Finally, in order to complete the justification of the almost equivalence relation (A.1), we will prove the lemma A.2 as follows.

Proof of lemma A.2. To prove (A.5), let us first recall from the definition of $H_{\sigma,\delta}^{s,\gamma}$ that

$$(A.8) \quad \begin{cases} \delta(1+y)^{-\sigma} \leq \omega \leq \delta^{-1}(1+y)^{-\sigma} \\ \partial_y \omega \leq \delta^{-1}(1+y)^{-\sigma-1}, \end{cases}$$

so $\frac{\partial_y \omega}{\omega} \leq \delta^{-2}(1+y)^{-1}$, and hence, for any $k = 1, 2, \dots, s$,

$$\begin{aligned} \|(1+y)^\gamma g_k\|_{L^2} &\leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \|(1+y) \frac{\partial_y \omega}{\omega}\|_{L^\infty} \|(1+y)^{\gamma-1} \partial_x^k(u-U)\|_{L^2} \\ &\leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k(u-U)\|_{L^2} \end{aligned}$$

which is inequality (A.5).

Next, we are going to show (A.6). The main observation is that we can also rewrite $g_k := \omega \partial_x \left(\frac{\partial_x^k(u-U)}{\omega} \right)$. Thus, applying (A.8)₁ and part (ii) of lemma B.1, we have

$$(A.9) \quad \begin{aligned} \|(1+y)^{\gamma-1} \partial_x^k(u-U)\|_{L^2} &\leq \delta^{-1} \|(1+y)^{\gamma-\sigma-1} \frac{\partial_x^k(u-U)}{\omega}\|_{L^2} \\ &\leq C_{\gamma,\sigma} \delta^{-1} \left\{ \left\| \frac{\partial_x^k U}{\omega|_{y=0}} \right\|_{L^2(\mathbb{T})} + \left\| (1+y)^{\gamma-\sigma} \partial_y \left(\frac{\partial_x^k(u-U)}{\omega} \right) \right\|_{L^2} \right\} \\ &\leq C_{\gamma,\sigma} \delta^{-2} \{ \|\partial_x^k U\|_{L^2(\mathbb{T})} + \|(1+y)^\gamma g_k\|_{L^2} \}. \end{aligned}$$

Now, using triangle inequality, (A.8) and (A.9), we also have

$$(A.10) \quad \begin{aligned} \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} &\leq \|(1+y)^\gamma g_k\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k(u-U)\|_{L^2} \\ &\leq C_{\gamma,\sigma,\delta} \{ \|(1+y)^\gamma g_k\|_{L^2} + \|\partial_x^k U\|_{L^2(\mathbb{T})} \}. \end{aligned}$$

Summing up (A.9) and (A.10), we prove (A.6).

□

APPENDIX B. CALCULUS INEQUALITIES

In this appendix we will introduce several calculus inequalities for the incompressible velocity field (u, v) , the vorticity ω and the quantity g_k . These inequalities are not related to any equations; they hold just because of elementary computations.

B.1. Basic Inequalities. In this subsection we will state without proof two elementary inequalities (i.e., lemma B.1 and lemma B.2 below).

Let us first introduce Hardy's type inequalities.

Lemma B.1 (Hardy's Type Inequalities). *Let $f : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$. Then*

(i) *if $\lambda > -\frac{1}{2}$ and $\lim_{y \rightarrow +\infty} f(x, y) = 0$, then*

$$(B.1) \quad \|(1+y)^\lambda f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq \frac{2}{2\lambda+1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)};$$

(ii) *if $\lambda < -\frac{1}{2}$, then*

$$(B.2) \quad \|(1+y)^\lambda f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq \sqrt{-\frac{1}{2\lambda+1}} \|f|_{y=0}\|_{L^2(\mathbb{T})} - \frac{2}{2\lambda+1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}.$$

The proof of lemma B.1 is elementary, so we leave it for the reader.

Next, we will state the following Sobolev's type inequality.

Lemma B.2. *Let $f : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$. Then there exists a universal constant $C > 0$ such that*

$$(B.3) \quad \|f\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq C \{ \|f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_x f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_y^2 f\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \}$$

To prove lemma B.2, one may extend the domain of f to \mathbb{R}^2 via the standard extension argument. Then inequalities (B.3) follows easily by the Fourier's inversion formula. We leave this to the reader as well.

B.2. Estimates for $H_{\sigma,\delta}^{s,\gamma}$ Functions. In this subsection we will use the weighted norm $\|\cdot\|_{H_g^{s,\gamma}}$ to control certain L^2 and L^∞ norms of u, v, ω, g_k and their derivatives. To derive these estimates, we shall apply lemma B.1 and lemma B.2, which was introduced previously in subsection B.1.

Our aim is to prove the following

Proposition B.3 (L^2 and L^∞ Controls on u, v, ω and g_k). *Let the vector field (u, v) defined on $\mathbb{T} \times \mathbb{R}^+$ satisfy the incompressibility condition $\partial_x u + \partial_y v = 0$, the Dirichlet boundary condition $u|_{y=0} = v|_{y=0} \equiv 0$ and $\lim_{y \rightarrow +\infty} u = U$. If the vorticity $\omega := \partial_y u \in H_{\sigma,\delta}^{s,\gamma}$ for some constants $s \geq 4, \gamma \geq 1, \sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$, then we have the following estimates: there*

exists a constant $C_{s,\gamma,\sigma,\delta} > 0$ such that

Weighted L^2 Estimates:

(i) for all $k = 0, 1, \dots, s$,

$$(B.4) \quad \|(1+y)^{\gamma-1} \partial_x^k (u-U)\|_{L^2} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\},$$

(ii) for all $k = 0, 1, \dots, s-1$,

$$(B.5) \quad \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^2} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\},$$

(iii) for all $|\alpha| \leq s$,

$$(B.6) \quad \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^2} \leq \begin{cases} C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\} & \text{if } \alpha = (s, 0) \\ \|\omega\|_{H_g^{s,\gamma}} & \text{if } \alpha \neq (s, 0). \end{cases}$$

(iv) for all $k = 1, 2, \dots, s$,

$$(B.7) \quad \|(1+y)^\gamma g_k\|_{L^2} \leq \begin{cases} C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\} & \text{if } k = 1, 2, \dots, s-1 \\ \|\omega\|_{H_g^{s,\gamma}} & \text{if } k = s, \end{cases}$$

where the quantity $g_k := \partial_x^k \omega - \frac{\partial_y \omega}{\omega} \partial_x^k (u-U)$.

Weighted L^∞ Estimates:

(v) for all $k = 0, 1, \dots, s-1$,

$$(B.8) \quad \|\partial_x^k u\|_{L^\infty} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\},$$

(vi) for all $k = 0, 1, \dots, s-2$,

$$(B.9) \quad \left\| \frac{\partial_x^k v}{1+y} \right\|_{L^\infty} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\},$$

(vii) for all $|\alpha| \leq s-2$,

$$(B.10) \quad \|(1+y)^{\gamma+\alpha_2} D^\alpha \omega\|_{L^\infty} \leq C_{s,\gamma} \|\omega\|_{H_g^{s,\gamma}}.$$

Proof.

(i) It follows from the definition of $\|\cdot\|_{H^{s,\gamma-1}}$ that $\|(1+y)^{\gamma-1} \partial_x^k (u-U)\|_{L^2} \leq \|u-U\|_{H^{s,\gamma-1}}$, so inequality (B.4) is a direct consequence of the almost equivalence inequality (A.1).

(ii) Applying part (ii) of lemma B.1 and inequality (B.4), we have

$$\left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^2} \leq 2 \|\partial_x^{k+1} (u-U)\|_{L^2} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}$$

which is inequality (B.5).

(iii) Inequality (B.6) follows directly from the definition of $\|\omega\|_{H^{s,\gamma}}$ and inequality (A.1).

(iv) Since $\omega \in H_{\sigma,\delta}^{s,\gamma}$, we know that $\|(1+y)\frac{\partial_y w}{\omega}\|_{L^\infty} \leq \delta^{-2}$, so using triangle inequality, inequalities (B.4) and (B.6), we have

$$\|(1+y)^\gamma g_k\|_{L^2} \leq \|(1+y)^\gamma \partial_x^k \omega\|_{L^2} + \delta^{-2} \|(1+y)^{\gamma-1} \partial_x^k (u-U)\|_{L^2} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}$$

which is inequality (B.7) for $k = 1, 2, \dots, s-1$. When $k = s$, the better upper bound in (B.7) follows directly from the definition of $\|\omega\|_{H_g^{s,\gamma}}$.

(v) For any $k = 1, 2, \dots, s-1$, applying lemma B.2, inequalities (B.4) and (B.6), we have

$$\begin{aligned} \|\partial_x^k (u-U)\|_{L^\infty} &\leq C \{\|\partial_x^k (u-U)\|_{L^2} + \|\partial_x^{k+1} (u-U)\|_{L^2} + \|\partial_x^k \partial_y \omega\|_{L^2}\} \\ &\leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}, \end{aligned}$$

and hence, by the triangle inequality and $\|\partial_x^k U\|_{L^\infty} \leq C_s \|\partial_x^s U\|_{L^2}$, we justify (B.8).

For the case $k = 0$, let us first recall from the hypothesis that $\omega := \partial_y u > 0$, so

$$(B.11) \quad 0 \leq u \leq U = \int_0^{+\infty} \omega dy.$$

Thus, using Cauchy-Schwarz's inequality and estimate (B.6), we have

$$\|U\|_{L^2}^2 = \int_{\mathbb{T}} \left| \int_0^{+\infty} \omega dy \right|^2 dx \leq \frac{1}{2\gamma-1} \|(1+y)^\gamma \omega\|_{L^2}^2 \leq \frac{1}{2\gamma-1} \|\omega\|_{H_g^{s,\gamma}}^2,$$

and hence, by (B.11), Sobolev inequality and $\|\partial_x U\|_{L^2} \leq C_s \|\partial_x^s U\|_{L^2}$, we obtain

$$\|u\|_{L^\infty} \leq \|U\|_{L^\infty} \leq C \{\|U\|_{L^2} + \|\partial_x U\|_{L^2}\} \leq C_{s,\gamma,\sigma,\delta} \{\|\omega\|_{H_g^{s,\gamma}} + \|\partial_x^s U\|_{L^2}\}$$

which is inequality (B.8).

(vi) Applying triangle inequality, lemma B.2, $\partial_x u + \partial_y v = 0$ and $\omega = \partial_y u$, we have

$$\begin{aligned} \left\| \frac{\partial_x^k v}{1+y} \right\|_{L^\infty} &\leq \left\| \frac{y \partial_x^{k+1} U}{1+y} \right\|_{L^\infty} + \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^\infty} \\ &\leq C \left\{ \|\partial_x^{k+1} U\|_{L^2(\mathbb{T})} + \|\partial_x^{k+2} U\|_{L^2(\mathbb{T})} + \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{1+y} \right\|_{L^2} + \left\| \frac{\partial_x^{k+1} v + y \partial_x^{k+2} U}{1+y} \right\|_{L^2} \right. \\ &\quad \left. + \left\| \frac{\partial_x^k v + y \partial_x^{k+1} U}{(1+y)^3} \right\|_{L^2} + \left\| \frac{\partial_x^{k+1} (u-U)}{(1+y)^2} \right\|_{L^2} + \left\| \frac{\partial_x^{k+1} \omega}{1+y} \right\|_{L^2} \right\} \end{aligned}$$

which implies inequality (B.9) because of Wirtinger's inequality and (B.4) - (B.6).

(vii) Inequality (B.10) follows directly from lemma B.2 and inequality (B.6).

□

B.3. Estimates for Functions Vanishing at Infinity. In this subsection we will first define certain weighted norms involving time and spatial derivatives. Then we will state two basic inequalities about these norms, see proposition B.5 below. Finally, in proposition B.6

we will control the weighted norms of u and v by that of ω provided that $u - U$ and its derivatives vanish at $y = +\infty$. The vanishing hypotheses (i.e., the decay rates) are usually guaranteed by proposition C.1 in applications.

Let us begin by defining the weighted norms.

Definition B.4. For any $s' \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, we define

$$\begin{aligned} ||| \cdot |||_{s', \gamma}^2 &:= \sum_{l=0}^{\lfloor \frac{s'}{2} \rfloor} \|\partial_t^l \cdot\|_{H^{s'-2l, \gamma}(\mathbb{T} \times \mathbb{R}^+)}^2, & ||| \cdot |||_{s'}^2 &:= \sum_{l=0}^{\lfloor \frac{s'}{2} \rfloor} \|\partial_t^l \cdot\|_{H^{s'-2l}(\mathbb{T})}^2, \\ ||| \cdot |||_{s', \infty, \gamma}^2 &:= \sum_{l=0}^{\lfloor \frac{s'}{2} \rfloor} \sum_{|\alpha| \leq s-2l} \|(1+y)^{\gamma+\alpha_2} \partial_t^l D^\alpha \cdot\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}^2 & \text{and} \\ ||| \cdot |||_{s', \infty}^2 &:= \sum_{l=0}^{\lfloor \frac{s'}{2} \rfloor} \|\partial_t^l \cdot\|_{W^{s'-2l, \infty}(\mathbb{T})}^2 \end{aligned}$$

where $\lfloor \frac{s'}{2} \rfloor$ denotes the largest integer which is less than or equal to $\frac{s'}{2}$.

Using Hölder inequality and Sobolev inequality, one can easily show the following

Proposition B.5 (Basic Inequalities for Weighted Norms).

(i) For any $s' \in \mathbb{N}$ and $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma = \gamma_1 + \gamma_2$,

$$|||F_1 F_2|||_{s', \gamma} \leq C_{s'} |||F_1|||_{s', \infty, \gamma_1} |||F_2|||_{s', \gamma_2}.$$

(ii) For any $s' \geq 5$ and $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma = \gamma_1 + \gamma_2$,

$$|||F_1 F_2|||_{s', \gamma} \leq C_{s', \gamma_1, \gamma_2} \{ |||F_1|||_{s'-1, \gamma_1} |||F_2|||_{s', \gamma_2} + |||F_1|||_{s', \gamma_1} |||F_2|||_{s'-1, \gamma_2} \}.$$

Next, we will state the weighted controls on u and v as follows.

Proposition B.6 (Weighted Controls on u and v). For any $s' \geq 4$ and $\gamma \geq 1$, let the vector field (u, v) defined on $\mathbb{T} \times \mathbb{R}^+$ satisfying the incompressibility condition $\partial_x u + \partial_y v = 0$, the Dirichlet boundary condition $v|_{y=0} = 0$ and $\lim_{y \rightarrow +\infty} \partial_t^l \partial_x^k u = \partial_t^l \partial_x^k U$ for all $l = 0, 1, \dots, \lfloor \frac{s'}{2} \rfloor$ and $k = 0, 1, \dots, s' - 2l + 1$. Denote the vorticity $\omega := \partial_y u$. Then there exists a universal constant $C > 0$ such that

$$(B.12) \quad |||u - U|||_{s', 0} \leq C |||\omega|||_{s', \gamma} \quad \text{and} \quad |||v + y \partial_x U|||_{s', -1} \leq C |||\partial_x \omega|||_{s', \gamma}.$$

Outline of the proof. The hypotheses of proposition B.6 allow us to apply lemma B.1 to $\partial_t^l \partial_x^k (u - U)$ and $\partial_t^l \partial_x^k (v + y \partial_x U)$ provided that $2l + k \leq s'$, so we obtain $|||u - U|||_{s', 0} \leq C |||\omega|||_{s', 1}$ and $|||v + y \partial_x U|||_{s', -1} \leq C |||\partial_x \omega|||_{s', 1}$ which imply (B.12) since $\gamma \geq 1$. \square

APPENDIX C. DECAY RATES FOR $H_{\sigma,\delta}^{s,\gamma}$ FUNCTIONS

The aim of this appendix is to prove that the actual pointwise decay rates of $H_{\sigma,\delta}^{s,\gamma}$ functions at $y = +\infty$ are better than the decay rates obtained by the Sobolev embeddings. The proof relies on a pointwise interpolation argument (see lemma C.3 below), which is a direct consequence of the Taylor's series expansion.

More specifically, we will prove the decay property of $D^\alpha \omega$ as y goes to $+\infty$ as follows.

Proposition C.1 (Decay Rates for $H_{\sigma,\delta}^{s,\gamma}$ Functions). *Let $s' \geq 4$ be an integer, $\gamma \geq 1, \sigma > \gamma + \frac{1}{2}$ and $\delta \in (0, 1)$. If $\omega \in H_{\sigma,\delta}^{s'+4,\gamma}$, then ω is $s' + 2$ times differentiable and there exists a constant $C_{s',\gamma,\delta,\|\omega\|_{H^{s'+4,\gamma}}} > 0$ such that for all $|\alpha| \leq s' + 2$,*

$$(C.1) \quad |D^\alpha \omega| \leq C_{s',\gamma,\delta,\|\omega\|_{H^{s'+4,\gamma}}} (1+y)^{-b_\alpha} \quad \text{in } \mathbb{T} \times \mathbb{R}^+$$

where the exponent

$$(C.2) \quad b_\alpha := \begin{cases} \sigma + \alpha_2 & \text{if } |\alpha| \leq 2 \\ \frac{\sigma + (2^{|\alpha|-2} - 1)\gamma}{2^{|\alpha|-2}} + \alpha_2 & \text{if } 2 \leq |\alpha| \leq s' + 1 \\ \gamma + \alpha_2 & \text{if } |\alpha| = s' + 2. \end{cases}$$

Remark C.2 (Decay Rates from Sobolev Embeddings). Using the standard Sobolev embedding argument and the definition of $H_{\sigma,\delta}^{s'+4,\gamma}$, one may prove that if $\omega \in H_{\sigma,\delta}^{s'+4,\gamma}$, then ω is $s' + 2$ times differentiable and there exists a constant $C_{s,\gamma} > 0$ such that

$$(C.3) \quad |D^\alpha \omega| \leq \begin{cases} \delta^{-1} (1+y)^{-\sigma-\alpha_2} & \text{if } |\alpha| \leq 2 \\ C_{s,\gamma} \|\omega\|_{H^{s'+4,\gamma}} (1+y)^{-\gamma-\alpha_2} & \text{if } 2 \leq |\alpha| \leq s' + 2. \end{cases}$$

Thus, the interesting part of proposition C.1 is the decay rate of (C.1) is better than that of (C.3). This slightly better pointwise decay will help us to deal with the boundary terms at $y = +\infty$ while we are integrating by parts in the y -direction (cf remark 5.7).

Proof of proposition C.1. According to remark C.2, we are only required to justify the inequality (C.1) with the decay rate defined in (C.2).

First of all, let us state without proof the following calculus lemma.

Lemma C.3 (Pointwise Interpolation). *Let $f : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be a twice differentiable function. Then we have the following:*

- (i) *If there exist constants C_0, C_2, b_0 and b_2 such that $|\partial_x^i f| \leq C_i (1+y)^{-b_i}$ for all $i = 0, 2$, then*

$$|\partial_x f| \leq 2\sqrt{C_0 C_2} (1+y)^{-\frac{b_0+b_2}{2}} \quad \text{in } \mathbb{T} \times \mathbb{R}^+.$$

- (ii) If there exist non-negative constants C_0, C_2, b_0 and b_2 such that $|\partial_y^i f| \leq C_i(1+y)^{-b_i}$ for all $i = 0, 2$, then

$$|\partial_y f| \leq 2\sqrt{C_0 C_2}(1+y)^{-\frac{b_0+b_2}{2}} \quad \text{in } \mathbb{T} \times \mathbb{R}^+.$$

The proof of lemma C.3 is based on the standard Taylor's series expansion technique, and will be omitted here.

Now, applying lemma C.3 to $D^\alpha \omega$ inductively on $|\alpha| = 3, 4, \dots, s' + 1$ with the inequality (C.3), we prove (C.1) with the exponent b_α defined in (C.2). \square

Remark C.4 (Further Improvement on the Decay Rate). The decay rate b_α defined in (C.2) is obviously not optimal because one can apply the pointwise interpolation lemma C.3 again to further improve it. However, we do not intend to optimize it here. Indeed, repeatedly applying the pointwise interpolation lemma C.3, one may improve the decay rate b_α as

$$b_\alpha := \begin{cases} \sigma + \alpha_2 & \text{if } |\alpha| \leq 2 \\ \frac{(s' + 2 - |\alpha|)\sigma + (|\alpha| - 2)\gamma}{s'} + \alpha_2^- & \text{if } 3 \leq |\alpha| \leq s' + 1 \\ \gamma + \alpha_2 & \text{if } |\alpha| = s' + 2. \end{cases}$$

We leave this proof for the interested reader.

APPENDIX D. EQUATIONS FOR a^ϵ AND g_s^ϵ

In this appendix we will derive the evolution equations for $a^\epsilon := \frac{\partial_y \omega^\epsilon}{\omega^\epsilon}$ and $g_s^\epsilon := \partial_x^s \omega^\epsilon - a^\epsilon \partial_x^s (u^\epsilon - U)$ provided that $\omega^\epsilon > 0$, $(u^\epsilon, v^\epsilon, \omega^\epsilon)$ and (p^ϵ, U) satisfy (4.1) - (4.4). These derivations just follow from direct computations.

Equation for a^ϵ :

Differentiating the vorticity equation (4.3)₁ with respect to y once, we obtain

$$(D.1) \quad (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \partial_y \omega^\epsilon = \epsilon^2 \partial_x^2 \partial_y \omega^\epsilon + \partial_y^3 \omega^\epsilon - \omega^\epsilon \partial_x \omega^\epsilon + \partial_x u^\epsilon \partial_y \omega^\epsilon.$$

Using (D.1) and the vorticity equation (4.3)₁, we can compute

$$(D.2) \quad \begin{aligned} (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) a^\epsilon &= \frac{(\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \partial_y \omega^\epsilon}{\omega^\epsilon} - \frac{\partial_y \omega^\epsilon (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y) \omega^\epsilon}{(\omega^\epsilon)^2} \\ &= \epsilon^2 \left\{ \frac{\partial_x^2 \partial_y \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_x^2 \omega^\epsilon}{\omega^\epsilon} \right\} + \left\{ \frac{\partial_y^3 \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_y^2 \omega^\epsilon}{\omega^\epsilon} \right\} - \partial_x \omega^\epsilon + a^\epsilon \partial_x u^\epsilon. \end{aligned}$$

On the other hand, by direct differentiations only, one may check that

$$(D.3) \quad \begin{cases} \partial_x^2 a^\epsilon = \frac{\partial_x^2 \partial_y \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_x^2 \omega^\epsilon}{\omega^\epsilon} - 2 \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x a^\epsilon \\ \partial_y^2 a^\epsilon = \frac{\partial_y^3 \omega^\epsilon}{\omega^\epsilon} - a^\epsilon \frac{\partial_y^2 \omega^\epsilon}{\omega^\epsilon} - 2 a^\epsilon \partial_y a^\epsilon. \end{cases}$$

Substituting (D.3) into (D.2), we obtain an equation for a^ϵ :

$$(D.4) \quad (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) a^\epsilon = 2\epsilon^2 \frac{\partial_x \omega^\epsilon}{\omega^\epsilon} \partial_x a^\epsilon + 2a^\epsilon \partial_y a^\epsilon - g_1^\epsilon + a^\epsilon \partial_x U,$$

where $g_1^\epsilon := \partial_x \omega^\epsilon - a^\epsilon \partial_x (u^\epsilon - U)$.

Equation for g_s^ϵ : (Derivation of equation (5.31))

Differentiating the evolution equations for ω^ϵ and $u^\epsilon - U$ (i.e., equations (5.29)) with respect to x s times, we have

$$(D.5) \quad \left\{ \begin{aligned} & (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) \partial_x^s \omega^\epsilon + \partial_x^s v^\epsilon \partial_y \omega^\epsilon \\ & \quad = - \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^{s-j} u^\epsilon \partial_x^{j+1} \omega^\epsilon - \sum_{j=1}^{s-1} \binom{s}{j} \partial_x^{s-j} v^\epsilon \partial_x^j \partial_y \omega^\epsilon \\ & (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) \partial_x^s (u^\epsilon - U) + \partial_x^s v^\epsilon \omega^\epsilon \\ & \quad = - \sum_{j=0}^{s-1} \binom{s}{j} \partial_x^{s-j} u^\epsilon \partial_x^{j+1} (u^\epsilon - U) - \sum_{j=1}^{s-1} \binom{s}{j} \partial_x^{s-j} v^\epsilon \partial_x^j \omega^\epsilon \\ & \quad \quad - \sum_{j=0}^s \binom{s}{j} \partial_x^j (u^\epsilon - U) \partial_x^{s-j+1} U. \end{aligned} \right.$$

To eliminate the problematic term $\partial_x^s v^\epsilon$, we subtract $a^\epsilon \times (D.5)_2$ from $(D.5)_1$, and obtain

$$(D.6) \quad \begin{aligned} & (\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) g_s^\epsilon + \{(\partial_t + u^\epsilon \partial_x + v^\epsilon \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2) a^\epsilon\} \partial_x^s (u^\epsilon - U) \\ & = 2\epsilon^2 \partial_x^{s+1} (u^\epsilon - U) \partial_x a^\epsilon + 2\partial_x^s \omega^\epsilon \partial_y a^\epsilon - \sum_{j=0}^{s-1} \binom{s}{j} g_{j+1}^\epsilon \partial_x^{s-j} u^\epsilon \\ & \quad - \sum_{j=1}^{s-1} \binom{s}{j} \partial_x^{s-j} v^\epsilon \{\partial_x^j \partial_y \omega^\epsilon - a^\epsilon \partial_x^j \omega^\epsilon\} + a^\epsilon \sum_{j=0}^s \binom{s}{j} \partial_x^j (u^\epsilon - U) \partial_x^{s-j+1} U. \end{aligned}$$

Substituting (D.4) into (D.6), we obtain equation (5.31).

APPENDIX E. CLASSICAL MAXIMUM PRINCIPLES

The main purpose of this appendix is to state two classical maximum principles, which are useful in subsection 5.2, for parabolic equations.

The first lemma is the maximum principle for bounded solutions to parabolic equations.

Lemma E.1 (Maximum Principle for Parabolic Equations). *Let $\epsilon \geq 0$. If $H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}^+) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}^+)))$ is a bounded function which satisfies the differential inequality:*

$$\{\partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2\} H \leq f H \quad \text{in } [0, T] \times \mathbb{T} \times \mathbb{R}^+$$

where the coefficients b_1, b_2 and f are continuous and satisfy

$$(E.1) \quad \left\| \frac{b_2}{1+y} \right\|_{L^\infty([0,T] \times \mathbb{T} \times \mathbb{R}^+)} < +\infty \quad \text{and} \quad \|f\|_{L^\infty([0,T] \times \mathbb{T} \times \mathbb{R}^+)} \leq \lambda,$$

then for any $t \in [0, T]$,

$$(E.2) \quad \sup_{\mathbb{T} \times \mathbb{R}^+} H(t) \leq \max\{e^{\lambda t} \|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0, t]} \{e^{\lambda(t-\tau)} \|H(\tau)|_{y=0}\|_{L^\infty(\mathbb{T})}\}\}.$$

The proof of lemma E.1 is a direct application of the classical maximum principle. For the reader's convenience, we will outline its proof as follows.

Outline of the proof of lemma E.1. For any $\mu > 0$, let us define $\mathcal{H} := e^{-\lambda t} H - \mu \left\| \frac{b_2}{1+y} \right\|_{L^\infty} t - \mu \ln(1+y)$. Then one may check that for any $\tilde{t} \in (0, T]$,

$$\{\partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2 + (\lambda - f)\} \mathcal{H} < 0 \quad \text{in } [0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+,$$

so by the classical maximum principle for parabolic equations, we have

$$\max_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} \mathcal{H} \leq \max\{\|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0, \tilde{t}]} \{e^{-\lambda \tau} \|H(\tau)|_{y=0}\|_{L^\infty(\mathbb{T})}\}\}$$

provided that $R \geq \exp\left(\frac{1}{\mu} \|H\|_{L^\infty}\right) - 1$. Therefore, for any $(x, y) \in \mathbb{T} \times \mathbb{R}^+$, we have

$$\begin{aligned} H(\tilde{t}, x, y) - \mu \left\| \frac{b_2}{1+y} \right\|_{L^\infty} e^{\lambda \tilde{t}} \tilde{t} - \mu e^{\lambda \tilde{t}} \ln(1+y) \\ \leq \max\{e^{\lambda \tilde{t}} \|H(0)\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \max_{\tau \in [0, \tilde{t}]} \{e^{\lambda(\tilde{t}-\tau)} \|H(\tau)|_{y=0}\|_{L^\infty(\mathbb{T})}\}\} \end{aligned}$$

which implies (E.2) if we take the limit $\mu \rightarrow 0^+$ and replace the arbitrary time \tilde{t} by t . □

The second lemma is the lower bound estimate on bounded solutions for parabolic equations.

Lemma E.2 (Minimum Principle for Parabolic Equations). *Let $\epsilon \geq 0$. If $H \in C([0, T]; C^2(\mathbb{T} \times \mathbb{R}^+)) \cap C^1([0, T]; C^0(\mathbb{T} \times \mathbb{R}^+))$ is a bounded function with*

$$\kappa(t) := \min\left\{\min_{\mathbb{T} \times \mathbb{R}^+} H(0), \min_{[0, t] \times \mathbb{T}} H|_{y=0}\right\} \geq 0$$

and satisfies

$$\{\partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2\} H = f H$$

where the coefficients b_1, b_2 and f are continuous and satisfy (E.1), then for any $t \in [0, T]$,

$$(E.3) \quad \min_{\mathbb{T} \times \mathbb{R}^+} H(t) \geq (1 - \lambda t e^{\lambda t}) \kappa(t).$$

The proof of lemma E.2 is also standard and very similar to that of lemma E.1. We will outline it here for the reader's convenience as well.

Outline of the proof of lemma E.2. For any fixed $\tilde{t} \in [0, T]$ and $\mu > 0$, let us define $h := e^{-\lambda t} \{H - \kappa(\tilde{t})\} + \left\{ \lambda \kappa(\tilde{t}) + \mu \left\| \frac{b_2}{1+y} \right\|_{L^\infty} \right\} t + \mu \ln(1+y)$. Then one may check that

$$\{\partial_t + b_1 \partial_x + b_2 \partial_y - \epsilon^2 \partial_x^2 - \partial_y^2 + (\lambda - f)\} h > 0 \quad \text{in } [0, \tilde{t}] \times \mathbb{T} \times \mathbb{R}^+,$$

so by the classical maximum principle for parabolic equations, we have

$$\min_{[0, \tilde{t}] \times \mathbb{T} \times [0, R]} h \geq 0$$

provided that $R \geq \exp\left(\frac{1}{\mu} \{\|H\|_{L^\infty} + \kappa(\tilde{t})\}\right) - 1$. Taking the limit $R \rightarrow +\infty$, and then $\mu \rightarrow 0^+$, we obtain

$$H(\tilde{t}) \geq (1 - \lambda \tilde{t} e^{\lambda \tilde{t}}) \kappa(\tilde{t})$$

which implies inequality (E.3) if we replace the arbitrary time \tilde{t} by t .

□

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